Moderations Syllabus and Synopses 2009–2010
Honour School of Mathematics & Philosophy
for examination in 2010

Contents

1 Foreword

Syllabus

The syllabus here is that referred to in the *Examination Regulations 2009* **Special Regulations for Honour Moderations in Mathematics and Philosophy** (p.96-97).

Synopses

The synopses give some additional detail, and show how the material is split between the different lecture courses. They also include details of recommended reading.

Notice of misprints or errors of any kind, and suggestions for improvements in this booklet should be addressed to the Academic Assistant in the Mathematical Institute.

2 Syllabus

2.1 Pure Mathematics I

Naïve treatment of sets and mappings. Definition of a countable set. Relations, equivalence relations and partitions.

Vector spaces over $\mathbb{R}$ and $\mathbb{C}$; subspaces. The span of a (finite) set of vectors; spanning sets; examples. Finite dimensionality.

Algebra of matrices. The space of $m \times m$ matrices.

Linear dependence and linear independence. Definition of a basis; reduction of a spanning set and extension of a linearly independent set to a basis; proof that all bases have the same size. Dimension of the space. Co-ordinates with respect to a basis.

Elementary row operations on matrices; echelon form and row-reduction. Applications to finding bases of vector spaces. Matrix representation of a system of linear equations. Invariance of the row space under row operations; row rank.

Linear maps (also called linear transformations) from one (real) vector space to another. The matrix representation of a linear map with respect to fixed bases; change of basis and co-ordinate systems. Composition of linear maps and product of matrices.

Sums and intersections of subspaces; formula for the dimension of the sum.

The image and kernel of a linear map. The Rank-Nullity Theorem. Applications.

Significance of image, kernel, rank and nullity for systems of linear equations. Solution by Gaussian elimination. Bases of solution space of homogeneous equations.

Invertible matrices; use of row operations to decide invertibility and to calculate inverse. Column space and column rank. Equality of row rank and column rank.

Permutations, composition of permutations. Cycles, standard cycle-product notation. The parity of a permutation; calculation of parity from cycle structure.

Determinants of square matrices; properties of the determinant function. Computation of determinant by reduction to row echelon form. Relationship of determinant to invertibility. Determinant of a linear map of a vector space to itself.

Eigenvalues of linear maps of a vector space to itself. The characteristic polynomial of a square matrix; the characteristic polynomial of a linear map of a vector space to itself. The linear independence of a set of eigenvectors associated with distinct eigenvalues; diagonalisability of matrices.

Division algorithm and Euclid's algorithm for integers. Polynomials with real or complex coefficients, factorization; Division algorithm and Euclid's algorithm for polynomials.

Axioms for a group. Examples, including the symmetric groups. Subgroups, intersections of subgroups. The order of a subgroup. The order of an element. Cyclic subgroups. Cosets and Lagrange's Theorem. Straightforward examples. The structure of cyclic groups; orders of elements in cyclic groups.

Axioms for a Ring. Definition of a Field and an Integral Domain.

Homomorphisms of groups. The (First) Isomorphism Theorem.

Ring homomorphisms and their kernels; First Isomorphism Theorem for commutative rings.

Euclidean Geometry in two and three dimensions approached by vectors and co-ordinates. Vector algebra. Equations of planes, lines and circles. Rotations, reflections, isometries. Parametric representation of curves, tangents, conics (normal form only) focus, directrix. Simple surfaces: spheres, right circular cones.

2.2 Pure Mathematics II

Real numbers: arithmetic, ordering, suprema, infima; real numbers as a complete ordered field. The real numbers are uncountable.

The complex number system. De Moivre's Theorem, polar form, the triangle inequality.

Sequences of (real or complex) numbers. Limits of sequences of numbers; the algebra of limits. Order notation.

Subsequences; every subsequence of a convergent sequence converges to the same limit. Bolzano–Weierstrass Theorem. Cauchy's convergence principle. Limit point of a subset of the line or plane.

Series of (real or complex) numbers. Convergence of series. Simple examples to include geometric progressions and power series. Alternating series test, absolute convergence, comparison test, ratio test, integral test.

Power series, radius of convergence, important examples including exponential, sine and cosine series.

Continuous functions of a single real or complex variable. The algebra of continuous functions. A continuous real-valued function on a closed bounded interval is bounded, achieves its bounds and is uniformly continuous. Intermediate Value Theorem. Inverse Function Theorem for continuous strictly monotonic functions.

Sequences and series of functions. The uniform limit of a sequence of continuous functions is continuous. Weierstrass's M-test. Continuity of functions defined by power series. Definition of derivative of a function of a single real variable. The algebra of differentiable functions. Rolle's Theorem. Mean Value Theorem. L'Hôpital's rule. Taylor's expansion with remainder in Lagrange's form. Binomial theorem with arbitrary index.

Step functions and their integrals. The integral of a continuous function on a closed bounded interval defined as the supremum of the integral of the step functions it dominates. Properties of the integral including linearity and the interchange of integral and limit for a uniform limit of continuous functions on a bounded interval. The Mean Value Theorem for Integrals. The Fundamental Theorem of Calculus; integration by parts and substitution.

Term-by-term differentiation of a (real) power series (interchanging limit and derivative for a series of functions where the derivatives converge uniformly); relationships between exponential, trigonometric functions and hyperbolic functions.

The geometry of the complex numbers. The Argand diagram. Complex numbers as a method of studying plane geometry, roots of unity, cocyclic points, Ptolemy's Theorem, equations of lines and circles (Apollonius's Theorem). Möbius transformations acting on the extended complex plane. Möbius transformations take lines and circles to lines and circles, Stereographic projection. The Riemann sphere and the conformal map to the extended complex plane. The geometry of the sphere: great circles, triangles (their six angles, angle-excess and area formula). Platonic solids.

3 Synopses of Lectures

3.1 Introductory Courses to Pure Mathematics

There are three short introductory courses in Pure Mathematics within the first two weeks of Michaelmas term to help students adjust to University Mathematics. These are *Introduction to Pure Mathematics*, *Introduction to Complex Numbers* and *Reasoning and Proofs.*

3.1.1 Introduction to Pure Mathematics — Prof. Vaughan-Lee — 5 MT

There will be 5 introductory lectures in the first week of Michaelmas term.

Overview

The purpose of these introductory lectures is to establish some of the basic notation of mathematics, introduce the elements of (naïve) set theory and the nature of formal proof.

Learning Outcomes

Students will:

(i) have the ability to describe, manipulate, and prove results about sets and functions using standard mathematical notation;

(ii) know and be able to use simple relations;

(iii) have the ability to follow and to construct proofs by Mathematical Induction (including strong induction, minimal counterexample).

Synopsis

Sets: examples including the natural numbers, the integers, the rational numbers, the real numbers; inclusion, union, intersection, power set, ordered pairs and cartesian product of sets. Relations. Definition of an equivalence relation. Examples.

The well-ordering property of the natural numbers. Induction as a method of proof, including a proof of the binomial theorem with non-negative integral coefficients.

Maps: composition, restriction, 1-1ness, ontoness, invertible maps, images and preimages.

Reading

1. G. Smith, *Introductory Mathematics: Algebra and Analysis* (Springer-Verlag, London, 1998), Chapters 1 and 2.

2. Robert G. Bartle, Donald R. Sherbert, *Introduction to Real Analysis* (Wiley, New York, Third Edition, 2000), Chapter 1 and Appendices A and B.

3. C. Plumpton, E. Shipton, R. L. Perry, *Proof* (MacMillan, London, 1984).

3.1.2 Introduction to Complex Numbers — Dr Szendroi — 2 MT

This course will run in the first week of Michaelmas Term.

There is generally no tutorial for this course. Solutions to the problem sheet are posted on Monday of week 2 and students are asked to mark their problem sheet and notify their tutor.

Overview

This course aims to give all students a common background in complex numbers.

Learning Outcomes

Students will be able to:

(i) manipulate complex numbers with confidence;

(ii) understand geometrically their representation on the Argand diagram including the nth roots of unity;

(iii) know the polar representation form and be able to apply it.

Synopsis

Basic arithmetic of complex numbers, the Argand diagram; modulus and argument of a complex number. Statement of the Fundamental Theorem of Algebra. Roots of unity. De Moivre's theorem. Simple transformations in the complex plane. Polar form $r\mathrm{e}^{\mathrm{i}\theta}$, with applications.

Reading

1. R A Earl, *Bridging course material on complex numbers*

2. D W Jordan & P Smith, *Mathematical Techniques* (Oxford University Press, Oxford, 2002), Ch. 6.

3.1.3 Reasoning and Proofs — Dr Knight — 3 MT

This course will run in the second week of Michaelmas term.

Overview

Prior to their arrival, undergraduates are encouraged to read Professor Batty's study guide "How do undergraduates do Mathematics?" The aim of this course is to provide an initial treatment of this material to better prepare students for the Mathematics programme.

This course builds on the first course of Introduction to Pure Mathematics exposing students to further examples and applications.

Learning Outcomes

Students will begin to develop:

(i) a reflective approach to "How to do Mathematics," through theory and practice;

(ii) sound reasoning skills;

(iii) the ability to identify the essential parts of a proof technique, to construct proofs, setting them out clearly, concisely and accurately.

Synopsis

Beginning University Mathematics: Studying the theory, problem solving, writing mathematics. Examples from Applied Mathematics. [1 lecture]

Formulation of Mathematical statements with examples. Including hypotheses, conclusions, if, only if, if and only if, and, or. Quantifiers - for all, there exists. [1 lecture]

Proofs - standard techniques with examples. Counter examples. Constructing proofs. Experimentation, followed by making the proof precise. Example of proof by contradiction and induction. [1 lecture]

Reading

1. C. J. K. Batty, *How do undergraduates do Mathematics?* (Mathematical Institute Study Guide, 1994).
2. R. B. J. T. Allenby, *Numbers and Proofs* (Modular Mathematics Series), (Arnold, London, 1997).
3. R. A. Earl, *Bridging Material on Induction.*
4. G. Pólya. *How to solve it : a new aspect of mathematical method* (Penguin, 1990).

4 Pure Mathematics I

4.1 Linear Algebra I — Dr Erdmann — 14 MT

Overview

Linear algebra pervades and is fundamental to algebra, geometry, analysis, applied mathematics, statistics, and indeed most of mathematics. This course lays the foundations, concentrating mainly on vector spaces and matrices over the real and complex numbers. The course begins with examples focussed on $\mathbb{R}^2$ and $\mathbb{R}^3$, and gradually becomes more abstract.

Learning Outcomes

Students will:

(i) understand the notions of a vector space, a subspace, linear dependence and independence, spanning sets and bases within the familiar setting of $\mathbb{R}^2$ and $\mathbb{R}^3$;

(ii) understand and be able to use the abstract notions of a general vector space, a subspace, linear dependence and independence, spanning sets and bases and be able to prove results related to these concepts;

(iii) have an understanding of matrices and of their applications to the algorithmic solution of systems of linear equations and to their representation of linear maps of vector spaces.

Synopsis

Algebra of matrices.

Vector spaces over the real or complex numbers; examples. Subspaces; examples. Linear combinations of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. Lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$. Linear independence and Linear dependence. The span of a (finite) set of vectors; spanning sets. Examples.

The space of $m \times m$ matrices.

Elementary row operations on matrices; echelon form and row-reduction. Matrix representation of a system of linear equations. Invariance of the row space under row operations; row rank.

Definition of a basis; application of elementary row operations to finding basis of a vector space. Reduction of a spanning set and extension of a linearly independent set to a basis; proof that all bases have the same size, the dimension of a vector space. Co-ordinates with respect to a basis.

Sums and intersections of subspaces; formula for the dimension of the sum.

Linear maps (or transformations); examples. The image and kernel of a linear map. The Rank-Nullity Theorem. Applications.

The matrix representation of a linear transformation with respect to fixed bases. Change of basis and co-ordinate systems. Composition of linear maps and the product matrices.

Significance of image, kernel, rank and nullity for systems of linear equations including geometrical examples. Solution by Gaussian elimination. Bases of solution space of homogeneous equations. Applications to finding bases of vector spaces.

Invertible matrices; use of row operations to decide invertibility and to calculate inverse.

Column space and column rank. Equality of row rank and column rank.

[Note : the terms *linear map* and *linear transformation* have the same meaning.]

Main Text

T. S. Blyth and E. F. Robertson, *Basic Linear Algebra* (Springer, London, 1998).

Further Reading

1. C. W. Curtis, *Linear Algebra - An Introductory Approach* (Springer, London, 4th edition, reprinted 1994).
2. R. B. J. T. Allenby, *Linear Algebra* (Arnold, London, 1995).
3. D. A. Towers, *A Guide to Linear Algebra* (Macmillan, Basingstoke, 1988).
4. D. T. Finkbeiner, *Elements of Linear Algebra* (Freeman, London, 1972). [Out of print, but available in many libraries.]
5. B. Seymour Lipschutz, Marc Lipson, *Linear Algebra* (McGraw Hill, London, Third Edition, 2001).
6. R. B. J. T. Allenby, *Rings, Fields and Groups* (Edward Arnold, London, Second Edition, 1999). [Out of print, but available in many libraries also via Amazon.]

4.2 Linear Algebra II — Dr Neumann — 8HT

Learning Outcomes

Students will:

(i) have an understanding of matrices, and the representation of linear maps between vector spaces, including a change of basis;

(ii) understand the elementary properties of determinants;

(iii) understand the elementary parts of eigenvalue theory with some of its applications.

Synopsis

Review of a matrix of a linear maps with respect to bases, and change of bases. [1 lecture]

Permutations of a finite set, composition of permutations. Cycles and standard cycle-product notation. The result that a permutation is a product of transpositions. The parity of a permutation; it is well-defined; its relation to cycle structure.

Determinants of square matrices; properties of the determinant function; determinants and the scalar triple product. [2 lectures]

Computation of determinant by reduction to row echelon form.

Proof that a square matrix is invertible if and only if its determinant is non-zero.

Determinant of a linear maps of a vector space to itself. [$2\frac{1}{2}$ lectures]

Eigenvalues and eigenvectors of linear maps from a vector space to itself. The characteristic polynomial of a square matrix; the characteristic polynomial of a linear map from a vector space to itself. The linear independence of a set of eigenvectors associated with distinct eigenvalues; diagonalisability of matrices. Applications to solving linear systems of first order differential equations. [$2\frac{1}{2}$ lectures]

Reading List

1. C. W. Curtis, *Linear Algebra - An Introductory Approach* (Springer, New York, 4th edition, reprinted 1994).
2. R. B. J. T. Allenby, *Linear Algebra* (Arnold, London, 1995).
3. T. S. Blyth and E. F. Robertson, *Basic Linear Algebra* (Springer, London 1998).
4. D. A. Towers, *A Guide to Linear Algebra* (Macmillan, Basingstoke 1988).
5. D. T. Finkbeiner, *Elements of Linear Algebra* (Freeman, San Francisco, 1972). [Out of print, but available in many libraries]
6. S. Lang, *Linear Algebra* (Springer, London, Third Edition, 1987).

   For permutations:
7. R. B. J. T. Allenby, *Rings, Fields and Groups* (Edward Arnold, London, Second Edition, 1999). [Out of print, but available in many libraries also via Amazon.]
8. W.B.S. Stewart, *Abstract Algebra*, Mathematical Institute Notes (1994).

4.3 An Introduction to Groups, Rings, and Fields — Prof. Priestley — 8 HT and 8 TT

Overview

Abstract algebra evolved in the twentieth century out of nineteenth century discoveries in algebra, number theory and geometry. It is a highly developed example of the power of generalisation and axiomatisation in mathematics. The first objective is to develop some concepts of set theory as a useful language for abstract mathematics. Next we aim to introduce the student to some of the basic techniques of algebra. Thirdly, we aim to introduce group theory through axioms and examples both because groups are typical and famous examples of algebraic structures, and because of their use in the measurement of symmetry. Finally we begin the study of fields and rings which are important in the study of polynomials and their zeros.

Learning Outcomes

Students will have developed an abstract approach to reasoning about number systems, their arithmetic structures and properties. They will be familiar with the axioms of a group, ring, and field together with examples. They will have a detailed knowledge of the ring of integers and the ring of polynomials and the Euclidean Algorithm in each case. They will have developed an appreciation of the isomorphism theorem for groups and rings.

Synopsis

HT (8 lectures)

Review of algebraic structures encountered in the course so far: $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$; $\mathbb{R}^n$ and $M_{m,n}(\mathbb{R})$.

Concept of a binary operation. Axioms for a group. Examples from above, including invertible $n \times n$ matrices under multiplication and fact that a vector space is an abelian group under $+$; the bijections of a set onto itself form a group under composition.

Axioms for a ring (not assumed commutative with identity). Optional additions: $\cdot$ commutative; $\exists$ identity for multiplication. Examples. Definition of a field and an integral domain, with preliminary examples. Fact that a field is an integral domain and result that a finite integral domain is a field.

[2 1/2 lectures]

The integers as an example of an integral domain which is not a field. Division algorithm and Euclidean algorithm. Statement of Fundamental Theorem of Arithmetic. Polynomials with real or complex coefficients; ring structure, degree of a polynomial; Division algorithm, Euclid's algorithm; the Remainder Theorem.

[2 1/2 lectures]

Equivalence relations: relationship to partitions; examples, including integers mod n.

[1 lecture]

The permutation group S_n: brief review of permutations of a finite set; decomposition of a permutation into a product of disjoint cycles.

Groups in general, with additional (straightforward) examples. Subgroups, with examples; subgroup of a cyclic group is cyclic. Products of groups.

[2 lectures]

TT (8 Lectures)

Brief review of material on groups from HT course.

Cosets and Lagrange's theorem; simple examples. The order of an element.

[2 lectures]

Homomorphisms of groups with motivating examples. Kernels, normal subgroups, The 1st Isomorphism Theorem. More on S_n: conjugacy in S_n and relationship to normality.

[4 lectures]

Ring homomorphisms and their kernels; examples. Ideals in commutative rings, 1st Isomorphism Theorem for commutative rings; the example of integers mod n.

[2 lectures]

Remark : The number of lectures is an indicative guide.

Reading List

1. R.B.J.T. Allenby *Rings, Fields and Groups* (Second Edition, Edward Arnold, 1999) [Out of print, but available in many libraries also via Amazon.]
2. I. N. Herstein, *An Introduction to Abstract Algebra.* (Third edition, Wiley, 1996), ISBN - 978-0-471-36879-3.

Alternative Reading

1. N. L. Biggs, *Discrete Mathematics* (OUP, revised edition).
2. Peter J. Cameron, *Introduction to Algebra*, (OUP, 1998), §§1.1, 1.2, 1.3, 1.4, 3.1, 3.2, 3.3, 3.4.
3. John B. Fraleigh, *A First Course in Abstract Algebra* (Seventh edition, Pearson, 2002).
4. W. Keith Nicholson, *Introduction to Abstract Algebra* (Second edition, John Wiley, 1999), §§1.2, 1.3, 1.4, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10.
5. Joseph J. Rotman, *A First Course in Abstract Algebra* (Second edition, Prentice-Hall, 2000), §§1.1, 1.2, 1.3, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 3.3, 3.5.
6. G. C. Smith, *Introductory Mathematics: Algebra and Analysis*, Springer Undergraduate Mathematics Series (1998), Chapters 1, 2, 5.
7. Joseph Gallian *Contemporary Abstract Algebra* (Houghton Mifflin, 2006) ISBN 0618122141 [probably better as a reference].

4.4 Geometry I — Dr Szendroi — 7 MT

Overview

These lectures give an introduction to elementary ideas in the geometry of Euclidean space through vectors.

Learning Outcomes

By the end of the course students will have a detailed knowledge of Euclidean geometry in two and three dimensions.

Synopsis

Euclidean geometry in two and three dimensions approached by vectors and co-ordinates. Vector addition and scalar multiplication. The scalar product, equations of planes, lines and circles. The vector product in three dimensions. Scalar triple products and vector triple products, vector algebra. Rotations, reflections, isometries. Parametric representation of curves, tangents; conics (normal form only), focus and directrix. Simple surfaces: spheres, right circular cones.

Reading

Main text

1. J. Roe, *Elementary Geometry* (OUP, 1993), Chapters 1, 2.2, 3.4, 4, 7.1, 7.2, 8.1-8.3.

Alternative Reading

1. M. Reid and B. Szendroi, *Geometry and Topology* (Cambridge University Press, 2005), Chapter 1.
2. D. A. Brannan, M. F. Esplen and J. J. Gray, *Geometry* (CUP, 1999), Chapter 1.
3. P. M. Cohn, *Solid Geometry* (Routledge and Kegan Paul, 1961), Chapters 1-3.
4. R. Fenn, *Geometry* (Springer, 2001), Chapters 2, 5, 7.
5. J. E. Marsden and A. J. Tromka, *Vector Calculus* (McGraw-Hill, fourth edition, 1996), Chapters 1, 2.
6. J. Silvester, *Geometry, Ancient and Modern* (OUP, 2001), Chapters 3, 5, 6, 7.

5 Pure Mathematics II

Students will already be aware of the importance and power of calculus. The aim of Analysis I, II and III is to put precision into its formulation and to make students more secure in their understanding and use. The lectures on analysis come in three parts, beginning in the first term with the properties of the real numbers and studying limits through sequences and series. The second part is concerned with continuity and differentiation, and the third part with integration. Power series and the functions they define provide a unifying theme.

5.1 Analysis I: Sequences and Series — Dr Earl — 14 MT

Overview

In these lectures we study the real and complex numbers and study their properties, particularly completeness; define and study limits of sequences, convergence of series, and power series.

Learning Outcomes

Student will have:

(i) an ability to work within an axiomatic framework.

(ii) a detailed understanding of how Cauchy's criterion for the convergence of real and complex sequences and series follows from the completeness axiom for $\mathbb{R}$, and the ability to explain the steps in standard mathematical notation.

(iii) knowledge of some simple techniques for testing the convergence of sequences and series, and confidence in applying them.

(iv) familiarity with a variety of well-known sequences and series, with a developing intuition about the behaviour of new ones.

(v) an understanding of how the elementary functions can be defined by power series, with an ability to deduce some of their easier properties.

Synopsis

Real numbers: arithmetic, ordering, suprema, infima; the real numbers as a complete ordered field. Definition of a countable set. The countability of the rational numbers. The reals are uncountable. The complex number system. The triangle inequality.

Sequences of real or complex numbers. Definition of a limit of a sequence of numbers. Limits and inequalities. The algebra of limits. Order notation: O, o.

Subsequences; a proof that every subsequence of a convergent sequence converges to the same limit; bounded monotone sequences converge. Bolzano-Weierstrass Theorem. Limit point of a set. Cauchy's convergence principle.

Series of real or complex numbers. Convergence of series. Simple examples to include geometric progressions and some power series. Absolute convergence, comparison test, ratio test, integral test. Alternating series test.

Power series, radius of convergence; important examples to include the exponential, cosine and sine series.

Reading – Main texts

1. Robert G. Bartle, Donald R. Sherbert, *Introduction to Real Analysis* (Wiley, Third Edition, 2000), Chapters 2, 3, 9.1, 9.2.
2. R. P. Burn, *Numbers and Functions, Steps into Analysis* (Cambridge University Press, 2000), Chapters 2-6. [This is a book of problems and answers, a DIY course in analysis.]
3. J. M. Howie, *Real Analysis*, Springer Undergraduate Texts in Mathematics Series (Springer, 2001) ISBN 1-85233-314-6.

Alternative Reading

The first five books take a slightly gentler approach to the material in the syllabus, whereas the last two cover it in greater depth and contain some more advanced material.

1. Mary Hart, *A Guide to Analysis* (MacMillan, 1990), Chapter 2.
2. J. C. Burkill, *A First Course In Mathematical Analysis* (CUP, 1962), Chapters 1, 2 and 5.
3. K. G. Binmore, *Mathematical Analysis, A Straightforward Approach* (Cambridge University Press, second edition, 1990), Chapters 1-6.
4. Victor Bryant, *Yet Another Introduction to Analysis* (Cambridge University Press, 1990), Chapters 1 and 2.
5. G. Smith, *Introductory Mathematics: Algebra and Analysis* (Springer-Verlag, 1998), Chapter 3 (introducing complex numbers).
6. Michael Spivak, *Calculus* (Benjamin, 1967), Parts I, IV, and V (for a construction of the real numbers).
7. Brian S. Thomson, Judith B. Bruckner, Andrew M. Bruckner, *Elementary Analysis* (Prentice Hall, 2001), Chapters 1-4.

5.2 Analysis II: Continuity and Differentiability — Dr Dyson — 16 HT

Overview

In this term's lectures we study continuity of functions of a real or complex variable, and differentiability of functions of a real variable.

Learning Outcomes

At the end of the course students will be able to apply limiting properties to describe and prove continuity and differentiability conditions for real and complex functions. They will be able to prove important theorems, such as the I.V.T., Rolle's and Mean Value, and will begin the study of power series and their convergence.

Synopsis

Definition of the function limit. Examples and counter examples to illustrate when $\lim_{x\to a} f(x) = f(a)$ (and when it doesn't). Definition of continuity of functions on subsets of $\mathbb{R}$ and $\mathbb{C}$ in terms of ϵ and δ. The algebra of continuous functions; examples, including polynomials. Continuous functions on closed bounded intervals: boundedness, maxima and minima, uniform continuity. Intermediate Value Theorem. Inverse Function Theorem for continuous strictly monotone functions. Sequences and series of functions. The uniform limit of a sequence of continuous functions is continuous. Weierstrass's M-test for uniformly convergent series of functions. Continuity of functions defined by power series. Definition of the derivative of a function of a real variable. Algebra of derivatives, examples to include polynomials and inverse functions. The derivative of a function defined by a power series is given by the derived series (proof not examinable). Vanishing of the derivative at a local maximum or minimum. Rolle's Theorem. Mean Value Theorem with simple applications: constant and monotone functions. Cauchy's (generalized) Mean Value Theorem and L'Hôpital's formula. Taylor's Theorem with remainder in Lagrange's form; examples of Taylor's Theorem to include the binomial expansion with arbitrary index.

Reading – Main texts

1. Robert G. Bartle, Donald R. Sherbert, *Introduction to Real Analysis* (Wiley, Third Edition, 2000), Chapters 4-8.

2. R. P. Burn, *Numbers and Functions, Steps into Analysis* (Cambridge University Press, 2000). [This is a book of problems and answers, a DIY course in analysis]. Chapters 6-9, 12.

3. Walter Rudin, *Principles of Mathematical Analysis* (McGraw-Hill, 3rd edition, 1976). Chapters 4,5,7.

4. J. M. Howie, *Real Analysis,* Springer Undergraduate Texts in Mathematics Series (Springer, 2001), ISBN 1-85233-314-6.

Alternative Reading

1. Mary Hart, *A Guide to Analysis* (MacMillan, 1990), Chapters 4,5.
2. J. C. Burkill, *A First Course in Mathematical Analysis* (CUP, 1962), Chapters 3, 4, and 6.
3. K. G. Binmore, *Mathematical Analysis* A Straightforward Approach, (CUP, second edition, 1990), Chapters 7-12, 14-16.
4. Victor Bryant, *Yet Another Introduction to Analysis* (CUP, 1990), Chapters 3 and 4.
5. M. Spivak, *Calculus* (Publish or Perish, 3rd Edition, 1994), Part III.
6. Brian S. Thomson, Judith B. Bruckner, Andrew M. Bruckner, *Elementary Analysis* (Prentice Hall, 2001), Chapters 5-10.

5.3 Analysis III: Integration — Prof. Haydon — 8 TT

Overview

In these lectures we define a simple integral and study its properties; prove the Mean Value Theorem for Integrals and the Fundamental Theorem of Calculus. This gives us the tools to justify term-by-term differentiation of power series and deduce the elementary properties of the trigonometric functions.

Learning Outcomes

At the end of the course students will be familiar with the construction of an integral from fundamental principles including important theorems. They will know when it is possible to integrate or differentiate term-by-term and be able to apply this to, for example, trigonometric series.

Synopsis

Step functions, their integral, basic properties. The application of uniform continuity to approximate continuous functions above and below by step functions. The integral of a continuous function on a closed bounded interval. Elementary properties of the integral of a continuous function: positivity, linearity, subdivision of the interval.

The Mean Value Theorem for Integrals. The Fundamental Theorem of Calculus; linearity of the integral, integration by parts and substitution.

The interchange of integral and limit for a uniform limit of continuous functions on a bounded interval. Term-by-term integration and differentiation of a (real) power series (interchanging limit and derivative for a series of functions where the derivatives converge uniformly); examples to include the derivation of the main relationships between exponential, trigonometric functions and hyperbolic functions.

Reading

1. T. Lyons *Lecture Notes.*

2. J. Roe, *Integration* Mathematical Institute Notes (1994).

3. H. A. Priestley, *Introduction to Integration* (Oxford Science Publications, 1997), Chapters 1–8. [These chapters commence with a useful summary of background 'cont and diff' and go on to cover not only the integration but also the material on power series.]

4. Robert G. Bartle, Donald R. Sherbert, *Introduction to Real Analysis* (Wiley, Third Edition, 2000), Chapter 8.

5.4 Geometry II — Dr Luke — 8 TT

Overview

The aim of these last 8 lectures in geometry is to introduce students to some of the most elegant and fundamental results in the geometry of the Euclidean plane and the sphere.

Learning Outcomes

By the end of the course students will be able to use complex numbers to represent simple geometric plane figures, to know important plane geometric theorems and be able to use Möbius transformations. They will know some elementary spherical geometry and the Platonic solids (their properties and why there are precisely five of them.)

Synopsis

Brief review of the geometry of the complex numbers, the Argand diagram. Complex numbers as a methods of studying plane geometry, examples to include roots of unity, cocyclic points, Ptolemy's Theorem. Equations of lines and circles (Apollonius's Theorem), Möbius transformations acting on the extended complex plane, stereographic projection. The classification of Möbius transformations as parabolic, elliptic, hyperbolic and loxodromic. Möbius transformations take lines and circles to lines and circles, The Riemann sphere and the conformal map to the extended complex plane.

The geometry of the sphere: great circles, distance between two points, triangles (their six angles, angle-excess and area formula, cosine rule for sides of strict spherical triangle).

Platonic solids and the corresponding subdivision of the sphere noting Euler's formula.

Reading

1. Roger Fenn, *Geometry* (Springer, 2001), Chapters 4 and 8.

2. J. Roe, *Elementary Geometry* (Oxford Science Publications, 1992), Section 6.5.

3. Liang-shin Hahn, *Complex Numbers and Geometry* (The Mathematical Association of America, 1994).

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