

Codimension-Two Free Boundary Problems



Keith Gillow
St Catherine's College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

November 1998

To my Parents
for their unending support and encouragement

Acknowledgements

I would like to thank Dr S.D. Howison and Dr J.R. Ockendon for their encouragement and guidance as my supervisors, Dr S.J. Chapman for all his advice and assistance and the Engineering and Physical Sciences Research Council for their continued support. I would also like to thank the DH9 crew and the computing support staff for providing several sanity preserving distractions. Finally I would like to thank my Parents and Kathryn for all their support, faith and interest.

Abstract

Over the past 30 years the study of free boundary problems has stimulated much work. However, there exists a widely occurring, but little studied subclass of free boundary problems in which the free boundary has dimension two fewer than that of the underlying space rather than the more commonly studied case of one less. These problems are called *codimension-two* free boundary problems.

In Chapter 1 the typical geometries required for such problems, the main mathematical techniques and the methodology used are discussed. Then, in Chapter 2, the techniques required to solve them are demonstrated using the particular example of the water entry problem. Further results for the water entry problem are then derived including an analysis of the relatively poorly understood water exit problem. In Chapter 3 a review is given of some classical contact and crack problems in solid mechanics. The inclusion of a cohesive zone in a dynamic type-III crack problem is considered. The Muskhelishvili potential method is presented and used to solve both a contact and crack problem. This enables the solution of a type-I crack problem relating to an ink delivery system to be found. In Chapter 4 a problem posed by car windscreen forming is addressed. A local solution near a corner is analysed to explain when and how point forces occur at the corners of the frame on which the simply supported windscreen rests. Then the full problem is solved numerically for different types of boundary condition. Chapters 5 and 6 deal with several sintering problems in viscous flow highlighting the value of the methodology introduced in Chapter 1. It will be shown how the Muskhelishvili potential method also carries over to Stokes flow problems. The difficulties of matching to an inner as opposed to an outer region are investigated. Lastly two interface problems between immiscible liquids are considered which show how the solution procedure is adapted when the field equation in the thin region is non-trivial. In the final chapter results are summarised, open problems listed and conclusions drawn.

Contents

1	Introduction	1
1.1	Typical geometry required for a codimension-two problem	2
1.2	Where codimension-two problems occur	3
1.2.1	Water entry	4
1.2.2	Viscous sintering	5
1.3	One-phase and two-phase problems	6
1.4	Outline of the general methodology	7
1.5	Mathematical techniques	8
1.5.1	Matched asymptotic expansions	8
1.5.1.1	A note on log matching	8
1.5.2	The Riemann problem	9
1.5.3	Variational formulations	10
1.6	Layout and aims	12
1.7	Statement of originality	14
2	The water entry problem	16
2.1	The two dimensional water entry model	17
2.1.1	The outer problem	18
2.1.1.1	The codimension-two free boundary problem	19
2.1.1.2	Determination of the surface elevation	23
2.1.1.3	The law of motion of the free point	23
2.1.2	The inner and jet regions	25
2.1.3	The leading order pressure and force on the body	25
2.2	Variational formulation of the outer problem	26
2.3	Stability and water exit	29
2.3.1	A local stability analysis of the water entry problem	29
2.3.1.1	The local space and time model	30
2.3.1.2	Comparison with other types of stability analysis	36

2.3.2	A linearised initial value problem	36
2.3.2.1	Determination of the velocity potential	38
2.3.2.2	Derivation of the free surface elevation	40
2.3.2.3	The law of motion of the free boundary	44
2.3.3	Water exit problems	46
2.3.4	Conjectures on how to pose an exit problem	47
2.3.5	The Basilisk lizard	47
2.4	Extensions to the model	48
2.4.1	Bodies initially in contact with the fluid	48
2.4.2	Non-constant body velocity	48
2.4.2.1	Three-dimensional axisymmetric problem	51
2.4.2.2	The ‘bouncing’ bomb	53
2.4.3	Non-symmetric body shape	54
3	Solid contact and crack problems	56
3.1	Equations of linear elasticity	60
3.1.1	Airy stress function	62
3.1.2	The Muskhelishvili potential	63
3.2	A dynamic type-III crack moving at a constant velocity	67
3.2.1	Solution when the crack face is stress free along its entire length	68
3.2.2	Inclusion of a cohesive zone	72
3.3	Two-dimensional contact of two identical elastic bodies	75
3.3.1	Normal contact force	76
3.3.1.1	Solution of the codimension-two model	78
3.3.1.2	Alternative solution procedure using the Muskhelishvili potential	82
3.3.1.3	Determination of the normal traction	84
3.3.1.4	Parameters determining the contact region	85
3.3.2	Application of a shearing force after initially applying a normal force	86
3.3.2.1	Determination of the traction with no slip	87
3.3.2.2	Determination of the traction with slip	88
3.3.3	Simultaneous variations of normal and shearing forces	90
3.4	A static type-I crack problem	91
3.4.1	An ink delivery problem	92

4	Contact of an elastic plate supported by a frame	100
4.1	Thin plate equations	101
4.1.1	Variational formulation	106
4.1.2	Nondimensional thin plate model	107
4.2	Behaviour near a corner	108
4.2.1	Asymptotics of the solution near a critical angle	113
4.3	Consideration of the problem for a rectangular plate	115
4.3.1	The codimension-two free boundary problem	116
4.3.1.1	Local analysis near a free point	117
4.3.1.2	Numerical solution	118
5	Stokes flow problems	127
5.1	Point sources in Stokes flow	130
5.1.1	The outer problem	134
5.1.2	The codimension-two problem	135
5.1.2.1	A local analysis at a free point	136
5.1.2.2	Solution of the codimension-two problem	138
5.1.2.3	Determination of the free surface	141
5.1.2.4	The law of motion of the free point	141
5.2	Stokes flow with non-zero surface tension	142
5.2.1	The full problem	142
5.2.2	The codimension-two problem	143
5.2.3	The inner region	144
5.2.4	Matching difficulties	145
5.2.5	Reassessment of the codimension-two region	146
5.2.6	Matching revisited	148
5.2.7	Minimising the effort and maximising the results	149
5.2.8	Extension to different initial free boundary shapes	151
5.3	Ice Closure	153
5.3.1	Rigid impermeable bed	154
5.3.2	Inclusion of sediment creep	156
6	Hele–Shaw problems	158
6.1	Point sources in a Hele–Shaw cell	159
6.2	The three discs problem	163
6.2.1	The first outer problem	164
6.2.2	The first codimension-two problem	165

6.2.2.1	Determination of the free surface position	167
6.2.2.2	The law of motion of the free point	167
6.2.3	The second outer problem	168
6.2.4	The second codimension-two region	169
6.2.4.1	Determination of the free surface position	171
6.2.4.2	The law of motion of the free point	171
6.2.5	The third outer problem	172
6.2.6	The third codimension-two region	172
6.2.6.1	Determination of the free surface position	174
6.2.6.2	The law of motion of the free point	174
6.2.7	Extension to n discs	175
6.3	Muskat problems	177
6.3.1	Problem 1: Circular source at infinity	180
6.3.2	Problem 2: Contact and non-contact regions interchanged . .	181
7	Conclusions and further work	183
7.1	Summary	183
7.2	General remarks	187
7.3	Two open problems	188
7.3.1	Hele-Shaw flow with non-zero surface tension	188
7.3.2	A note on inviscid sintering	190
7.4	Further work	191
A	Sobolev and Hilbert spaces	193
B	Riemann boundary value problems and index	197
B.1	The Riemann problem for a simply connected domain	198
B.1.1	Solution of the homogeneous problem	199
B.1.2	Solution of the non-homogeneous problem	201
B.2	Solution of the Riemann problem with discontinuous coefficients . . .	203
B.2.1	Reduction to a problem with continuous coefficients	204
B.2.2	Solution of the homogeneous problem	206
B.2.3	Solution of the non-homogeneous problem	207
B.3	The Riemann problem for open contours	208
B.4	Inversion of a Cauchy integral	212
B.5	The Riemann problem for a non-Hölder free term	213

C The Stokes flow sintering matching condition	215
C.1 Solution of Stokes flow with a cusp	215
C.1.1 Solution following the procedure suggested by Morgan	215
C.1.2 Direct method of obtaining the same result	219
D The Ivantsov parabola	220
Bibliography	224

List of Figures

1.1	Typical geometry leading to a two-dimensional codimension-two problem.	3
1.2	The water entry problem.	4
1.3	The Stokes flow viscous sintering problem.	5
1.4	The one-dimensional obstacle problem.	10
2.1	The geometry of the water entry problem.	16
2.2	The codimension-one water entry problem.	17
2.3	The codimension-two free boundary problem.	20
2.4	The free surface shape for a wedge.	24
2.5	The free surface shape for a parabola.	24
2.6	The perturbed problem for the velocity potential ϕ	37
2.7	The domain of definition of the free surface elevation for the water entry and water exit problems.	46
2.8	The position of the free point $d(t)$ in the case $d(0) = 0.5$	49
2.9	The free surface shape for a wedge initially in contact with the fluid.	49
2.10	The free surface shape for a dropped parabolic body.	51
3.1	The three types of solid mechanics problems to be considered.	56
3.2	Schematic of three types of crack.	58
3.3	The initial configuration of the crack.	68
3.4	The codimension-two type-III dynamic crack problem.	69
3.5	The codimension-two type-III dynamic crack problem in the moving frame.	70
3.6	The domain of the crack problem in the z , \tilde{s} and s planes.	71
3.7	The inner crack problem with a cohesive zone.	73
3.8	The positions of the deformed and undeformed bodies.	76
3.9	The codimension-two free boundary problem.	79
3.10	The geometry of the additional shearing force problem.	87
3.11	The type-I crack problem.	92

3.12	The industrial requirement.	93
3.13	The first aperture shape.	95
3.14	The second aperture shape.	97
3.15	The third aperture shape.	98
4.1	The geometry of a simplified plate glass problem.	100
4.2	An element of an elastic plate.	101
4.3	The moments and forces per unit length on the middle plane of an elastic plate.	102
4.4	Representation of how a point force occurs at the corner.	105
4.5	The simplified problem for a single corner.	108
4.6	The further simplified problem for a single corner.	109
4.7	The edge reaction for different values of α	113
4.8	The codimension-two laterally loaded plate problem.	116
4.9	The local problem near a free point.	117
4.10	The application of (4.20)–(4.40).	121
4.11	The points of reference and keys for Figures 4.12–4.14.	122
4.12	1×1 plate : (a) Displacement w , (b) Cross sections of the displacement w , (c) Reactions along the edge. See Figure 4.11 for keys.	123
4.13	1×2 plate : (a) Displacement w , (b) Cross sections of the displacement w , (c) Reactions along the edge. See Figure 4.11 for keys.	124
4.14	1×4 plate : (a) Displacement w , (b) Cross sections of the displacement w , (c) Reactions along the edge. See Figure 4.11 for keys.	125
5.1	The geometry of the two-point Stokes flow injection problem.	127
5.2	The free surface shape for the injection problem.	142
5.3	The free surface shape for the sintering problem.	149
5.4	The free surface shape for the sintering problem with initial shape $ x ^3$	153
5.5	The geometry of the ice closure problem.	153
5.6	The codimension-two ice closure problem.	154
5.7	The codimension-two ice and till closure problem.	156
6.1	The geometry of the two point Hele-Shaw injection problem.	159
6.2	The free surface shape for the injection problem.	162
6.3	The geometry of the three discs problem.	163
6.4	The full problem near the first codimension-two region.	165
6.5	The first codimension-two problem.	166

6.6	The free surface shape for the first codimension-two problem.	168
6.7	The full problem near the second codimension-two region.	169
6.8	The second codimension-two problem.	170
6.9	The free surface shape for the second codimension-two problem.	172
6.10	The full problem near the third codimension-two region.	173
6.11	The third codimension-two problem.	174
6.12	The free surface shape for the third codimension-two problem.	175
6.13	The Muskat problem.	178
6.14	The codimension-two Muskat problem.	179
6.15	The geometry of the second Muskat problem.	181
7.1	The inviscid sintering problem.	190
C.1	The geometry of the cusp problem.	216
C.2	The general geometry to explain the force at the cusp.	217
D.1	The Ivantsov parabola problem.	220
D.2	The Ivantsov parabola problem with two fluids.	222

Chapter 1

Introduction

The mathematical theory of free boundary problems has been extensively researched over the last thirty years. An indication of this can be seen from the vast array of material now available on the subject in the form of books [12, 17, 23, 67], proceedings [8, 10, 18, 31, 64, 68, 101] and bibliographies [13, 91]. Free boundary problems are defined as a set of differential equations which must be solved in a domain of dimension n whose boundary, of dimension $(n - 1)$, is unknown *a priori*. An example is the classical Stefan problem [12] of determining the position of the free boundary between melting ice and water. The boundaries between the regions of ice and water are known as *free boundaries* as they are free to lie anywhere in the domain and are only determined as part of the solution. As stated above the free boundary has one dimension fewer than that of the underlying domain and so the boundary could be termed the ‘codimension-one free boundary’. The inherent nonlinearity of these problems has prompted theoretical investigations into questions of existence, uniqueness, regularity of the boundary, numerical algorithms, stability and asymptotic behaviour.

The techniques that have been used to try to answer these and other questions include numerical analysis, asymptotic expansions and weak solutions. Asymptotic expansions involve the exploitation of small parameters which occur in the model under certain circumstances, whilst weak solutions, although rarely available, satisfy an integrated formulation of the problem and are not required to be as smooth as solutions to the original differential equations. When they exist, weak solutions can sometimes be closely related to the equally rare variational formulations of certain special free boundary problems and hence great unification can be achieved both analytically and numerically. At the opposite extreme, when very irregular or unstable free boundary morphology can occur in practice, none of these techniques is usually available even to demonstrate existence of the solution and much theoretical work remains to be done.

Despite all this mathematical activity, there remains a widely-occurring, but little-studied, subclass of free boundary problems in which the free boundary has dimension $(n - 2)$. Two early attempts at unification were made by Howison [35] and Ockendon [67] and more recently by Morgan [58] and together they have produced a review [36] in order to stimulate further discussion. The number of case studies of this type that have appeared in the literature is now very large. We have termed these problems *codimension-two* free boundary problems, since the free boundary has two fewer dimensions than that of the underlying space in which the field equations are to be solved. In a two-dimensional problem the codimension-two free boundary is a collection of ‘free points’, and in three dimensions it is a collection of ‘free curves’.

We remark that the term codimension-two free boundary problem is also applied to models of a free curve in \mathbb{R}^3 as is the case for fluid and superconducting vortex dynamics. However, in these cases the global attribute of the free boundary problem may be lost since to lowest order the dynamics of the free curve are often purely governed by a local problem. This thesis deals with a subset of codimension-two free boundary problems which are not of the vortex dynamics type but occur when the codimension-one free boundary is uniformly close to a known boundary. This will enable us to exploit a naturally occurring small parameter and often these problems will be susceptible to formulation as Riemann problems from which a solution can be systematically obtained.

1.1 Typical geometry required for a codimension-two problem

A codimension-two free boundary problem is usually (although not always — see Chapter 4) obtained as a particular limit of its corresponding codimension-one problem. To illustrate this point consider a codimension-one problem with a two dimensional geometry as in Figure 1.1. The curve $\Gamma(x, y, t) = 0$ is a prescribed boundary, that is its position is known. The position of the codimension-one free boundary $h(x, y, t) = 0$ which separates the two regions is unknown *a priori* and so must be solved for as part of the problem. In regions I and II, sets of partial differential equations are given which satisfy given boundary conditions on h and Γ , one more boundary condition being required on h than Γ in order to determine its position. For a codimension-two problem to arise, the free boundary must be uniformly close to the known boundary and the extent of the contact region II along the known boundary must be large compared to its width (their ratio is known as the aspect

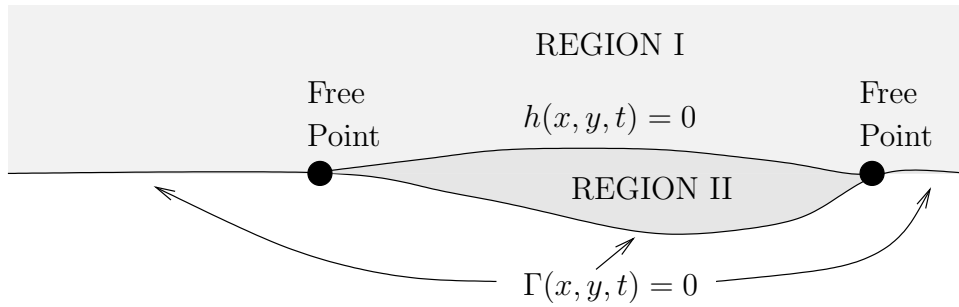


Figure 1.1: Typical geometry leading to a two-dimensional codimension-two problem.

ratio). The points at which the free boundary meets the fixed boundary are known as the free points. The majority of the problems considered in this thesis will only have two free points but, in general, there may be any number of them (of course the intersection of the free boundary with the prescribed boundary will be a curve in the three dimensional case).

If these conditions are met, the limit is taken as the aspect ratio tends to infinity. Then, any equations which formerly held in region I, which was of unknown extent, now hold in the known region ‘above’ $\Gamma(x, y, t) = 0$ and the only unknown parts of the geometry are the free points. The region of the fixed boundary between the free points is known as the ‘non-contact’ region because there is no contact between the fixed boundary and region I. Likewise, the region of the fixed boundary outside this region is known as the ‘contact’ region. This terminology is taken from that used in contact or obstacle problems in mechanics (such a problem is considered in Chapter 3).

1.2 Where codimension-two problems occur

A wide range of problems exhibit the characteristic type of geometry discussed above making them amenable to a codimension-two analysis. We list below just a few:

- Contact problems and crack problems in linear elasticity;
- Entry of a uniformly nearly flat body into a fluid (water entry) [47, 58, 102];
- Flow over a shallow step [70];
- Patch cavitation on a bluff head form [9, 36];
- Steady electropainting of a workpiece [58];

- Percolation in a sand bank [1];
- Viscous sintering of two cylinders under the action of surface tension [33, 36, 58].

Of the examples listed it is crack problems in solid mechanics that could be considered to be the progenitors of all codimension-two problems. We shall consider contact and crack problems in solid mechanics in more detail in Chapter 3. To expand upon the list, to give more of an idea of the type of problem that can be considered in a codimension-two framework, we shall now briefly describe two of those problems, the water entry problem and the viscous sintering problem.

1.2.1 Water entry

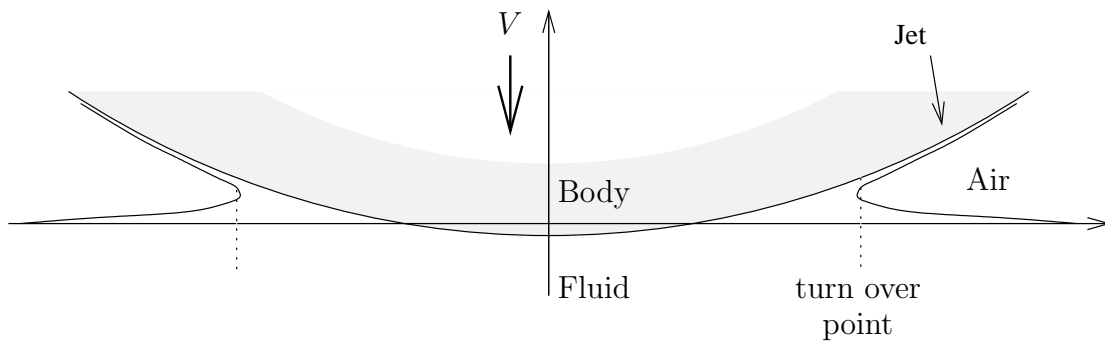


Figure 1.2: The water entry problem.

The water entry problem is the study of the normal impact of a nearly flat rigid convex body on an idealised fluid surface as shown in Figure 1.2. In terms of the discussion above region I is the fluid and region II is the air. The boundary of the body is known and the free boundary is the fluid surface. Since the body is nearly flat the free surface lies close to the body and to the undisturbed fluid level. Thus the problem has the required geometry described in Section 1.1 and both the free surface and fixed boundary can be linearised onto the prescribed undisturbed fluid level. One complication of this problem is the jets which are shown to form up the sides of the body. However, it has been shown [37] that these jets are thin and thus to leading order in the codimension-two problem they can be neglected such that to leading order the free surface does not turn over but rises to meet the body. The free points are where the free surface meets the body. The solution to this problem breaks down into three regions. There is the outer codimension-two region, this then drives inner regions around the turnover points, which in turn drive the thin jet regions. This problem will be reviewed in more detail in Chapter 2.

1.2.2 Viscous sintering

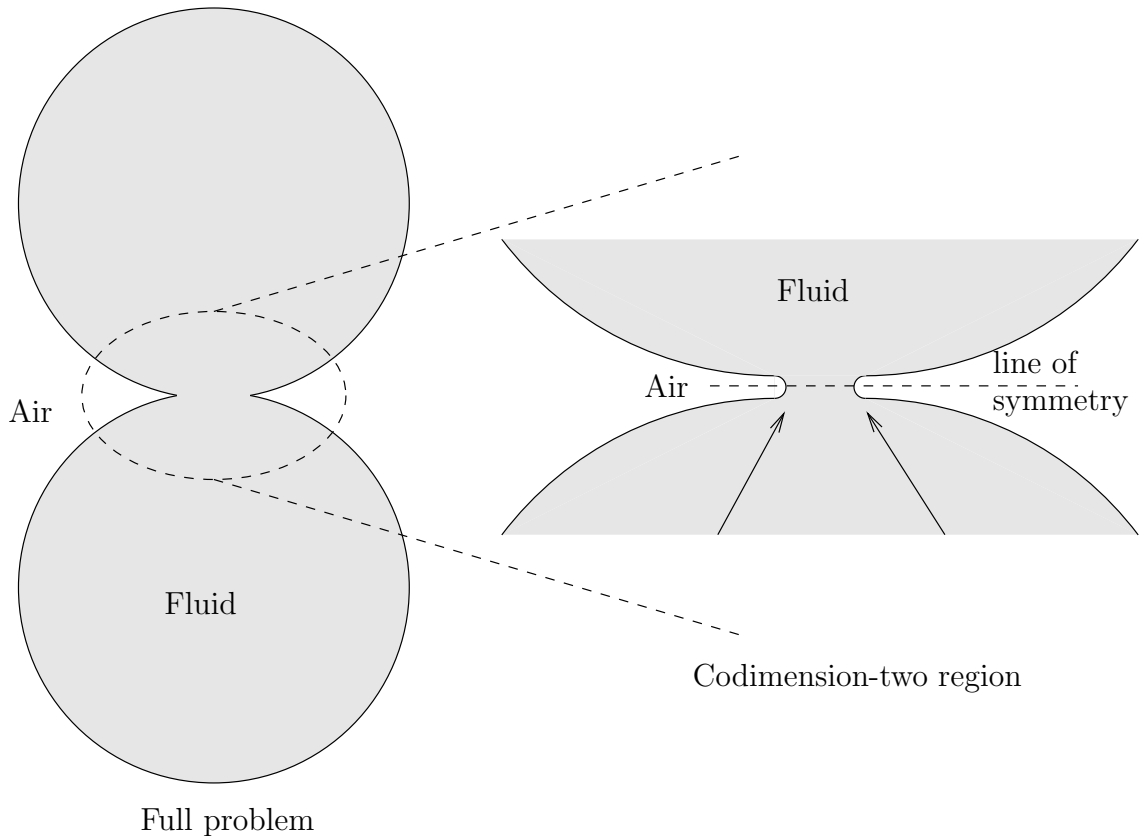


Figure 1.3: The Stokes flow viscous sintering problem.

Two identical cylinders of viscous fluid, which initially touch along a common generator, will coalesce purely under the action of surface tension to eventually form one large cylinder. This process is known as sintering and will be discussed in more detail in Chapter 5. For small times after the initial contact and in the region near to the contact the fluid flow problem can be formulated as a codimension-two problem. Region I for this problem is the fluid and region II is the air. The known boundary is the line of symmetry and the free boundary is the fluid boundary. For these small times and near to the contact region the free boundary is close to the line of symmetry. Hence in this region the problem exhibits the necessary geometry as described in Section 1.1. The free points are the points at which the fluid boundary meets the line of symmetry. This problem also exhibits three regions. However this problem is driven by the inner regions around the free points since the free boundary curvature is large. These inner regions then drive the codimension-two region indicated and then this drives an outermost region.

There are essentially two different ways in which codimension-two problems can arise. The first is as a local space and time analysis of a larger problem (such as in sintering). These problems often occur as the contact region grows, having initially zero extent. The codimension-two problem then gives a valuable insight into the nature of the solution at what is a crucial stage in the evolution. The second possibility is if the free boundary is close to the fixed boundary for all times. Such examples include the water entry problem (although the water entry problem can also occur as a local in time and space problem as will be discussed in Chapter 2).

1.3 One-phase and two-phase problems

In general the field equation will take a different form in regions I and II. If neither of the field equations in these regions is trivial, the problem is known as *two-phase*. In this case the field equation in region II has to be solved and substituted into the boundary conditions on the free boundary. The largeness of the aspect ratio can be exploited to simplify the field equation in this region enabling an approximate solution to be obtained. If the equations are trivial in region II (for example, it may be a vacuum), but not so in region I, the problem is known as *one-phase*.

Lastly, if the equations in region I are trivial, a different type of problem is realised which will not be considered in this thesis. The linearisation procedure in this case effectively causes region II to vanish. Since this was the only region with a non-trivial field equation there are now no equations remaining to solve. For this reason the problem is typically rescaled in such a way as to produce a partial differential equation for h the free surface shape. An example of this is the spread of a viscous drop on a solid surface where the free boundary thickness is described by a partial differential equation (for example, see [6]).

Generally, a codimension-one problem is nonlinear and cannot be solved analytically. Thus a major motivation for considering a codimension-two formulation of such a problem (even if only for a short time) is to obtain an approximate solution to an otherwise intractable problem. The codimension-two formulation has the key benefit that it is ‘only’ a mixed boundary value problem (admittedly over a domain where the position of the free points, at which the boundary conditions switch, are unknown) rather than a free boundary problem. Mixed boundary value problems are often quite well understood with well-developed techniques available for their solution, an example for Laplace’s equation being the theory of Riemann problems. Another

benefit to accrue from a codimension-two analysis for small times is the generation of accurate initial conditions for a full numerical solution which would otherwise have suffered from having to contend with ill-defined initial singularities.

1.4 Outline of the general methodology

The linearisation procedure described in Section 1.1 is only part of the whole solution procedure. As described above the codimension-two solution usually only represents the solution on some particular length scale and the whole problem can be broken down into several different regions. If the problem can be broken up in this way then we can follow the flow of information through the problem. Thus as described above for the sintering problem it is the innermost regions that drive the codimension-two region which in turn drives an outer region. The different regions of the solution must ‘match’ together, that is, for this flow of information, we solve the problem in the inner region and the solution when expanded in codimension-two region variables gives the matching condition (driving mechanism) for the codimension-two region. In turn expanding the codimension-two solution in outer variables gives the matching condition (driving mechanism) for the outer problem. A crucial step in solving these problems is to, therefore, identify how the information is flowing through the problem. According to the particular sequence of regions through which the information flows the solution procedure should be modified. However, as with any problem in asymptotics to say that the information is purely flowing in one direction would be misleading. In order to solve the problem in any one region assumptions are generally made which can only be verified at the matching stage. The assumptions we make are often made by applying Van Dyke’s maxim of taking the minimum allowable singularity at the free points. An example of this is when we solve the codimension-two water entry problem. The solution in the codimension-two region relies upon making certain assumptions about its behaviour near the free points. This assumption is only verified once the inner problem is solved and the two problems are found to match showing that the assumed behaviour was indeed correct.

The general procedure has been applied in a wide variety of physical situations, some of which are reviewed in [35, 36, 58]. The procedure can be broken down into three clear steps:

1. Identify the expected different regions of the solution and how the information will flow between them.

2. Solve the model in each region in turn following the flow of information using the solution to each previous region to drive the next one by means of the matching condition.
3. Perform any necessary matching between the regions to confirm the regions all match up and any assumptions that were made were correct.

1.5 Mathematical techniques

Three of the main techniques we will use in this thesis are matched asymptotic expansions, Riemann problems and variational formulations.

1.5.1 Matched asymptotic expansions

As discussed above the codimension-two solution is often only valid over some particular region of the full codimension-one problem, the codimension-one problem being broken down into two or more regions where the different regions of the solution must ‘match’ together. A key tool we will call upon is the Van Dyke Matching Principle [99]. Van Dyke’s matching principle can be written concisely as

$$mti(nto) = nto(mti) .$$

This notation is to be interpreted as follows:

- $mti(nto)$: Take the n -term outer expansion, write it in inner variables, and expand it to m -terms.
- $nto(mti)$: Take the m -term inner expansion, write it in outer variables, and expand it to n -terms.

Having calculated these two expressions they are written in common variables (either outer or inner). The above rule then states that the two expressions must be equal if they are to match. For a given $m = M$ and $n = N$ all possible matches must hold, for all combinations of $m = 1 \dots M$ and $n = 1 \dots N$, if we are to say the two expansions match up to these orders.

1.5.1.1 A note on log matching

Let us consider inner and outer expansions of the following function

$$f(x) = 1 + \frac{\log x}{\log \epsilon} .$$

Assume that $x = O(1)$ is the outer expansion and $x = \epsilon X$, $X = O(1)$ is the inner expansion, then the outer and inner expansions are simply

$$\begin{aligned} f_{out} &\sim 1 + \frac{\log x}{\log \epsilon} \\ f_{in} &\sim 2 + \frac{\log X}{\log \epsilon} . \end{aligned}$$

If we now apply the basic Van Dyke matching principle then we have

$$\begin{aligned} 1ti(1to) &= 1 \\ 1to(1ti) &= 2 \end{aligned}$$

which clearly shows the principle failing to work. But if we treat $\log \epsilon$ as $O(1)$ for the purposes of matching then all terms in both expansions above comprise the one term expansions and when we apply the matching principle everything now works as it must since they are simply expansions of one function on different scales.

Although in the limit $\epsilon \rightarrow 0$, $\epsilon^n \ll \epsilon^n \log 1/\epsilon$, which by the basic Van Dyke matching principle would say that they match at different orders, the above suggests that we should in fact consider them at the same time. In essence it suggests one should interpret the $\log \epsilon$ as being $O(1)$ for the purposes of applying the matching principle. This modification of the principle is discussed in more detail by Fraenkel [21].

1.5.2 The Riemann problem

We will use Riemann problems time and time again in this thesis as a means of formulating the codimension-two problem and hence finding the solution. In its general form a Riemann problem is to find an analytic function $\Phi(z)$, where $z = x + iy$, which satisfies the condition

$$\Phi^+(t) - G(t)\Phi^-(t) = g(t)$$

on L , a closed smooth curve where t is a variable denoting the position on L . The curve L divides the complex plane, on one side Φ takes the limiting value Φ^+ and on the other Φ^- . The function $G(t)$ is called the coefficient of the Riemann problem and $g(t)$ is called the free term. In the simplest form $G(t)$ and $g(t)$ satisfy Hölder conditions. The theory of the Riemann problem is discussed in detail in Appendix B in which the solution for the above problem is derived along with more complicated cases such as when G or g have discontinuities or when the curve L is open.

1.5.3 Variational formulations

As mentioned earlier, variational formulations are rare, but when they exist they can be extremely useful both analytically and numerically. Analytically they can often be used to prove uniqueness and existence of a solution. Numerically they can be used for a finite element solution. A simple example would be the one-dimensional obstacle problem shown in Figure 1.4.

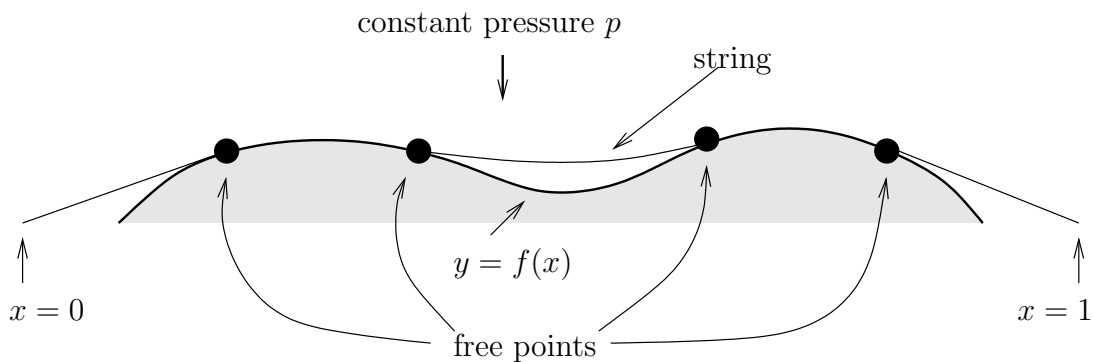


Figure 1.4: The one-dimensional obstacle problem.

This is the problem of determining the shape of an elastic string under constant pressure $p > 0$ when it is stretched over a rigid body $y = f(x)$. On the non-contact region the displacement is governed by the equation for an elastic string under constant load, whilst on the contact region the displacement coincides with the body shape. At the free points the contact is smooth (if it were not it would lead to an infinite force on a point). At the end points the string is fixed. Lastly, we have two inequalities which state that the string lies above the rigid body and the downward force per unit length on the string is not greater than p . In summary this gives the model

$$\begin{aligned}
 u_{xx} &= p && \text{on the non-contact region} \\
 u &= f && \text{on the contact region} \\
 u &= f && \text{at the free points} \\
 u_x &= f_x && \text{at the free points} \\
 u &\geq f && \text{for } x \in [0, 1] \\
 u_{xx} &\leq p && \text{for } x \in [0, 1] \\
 u &= 0 && \text{at } x = 0, 1 \text{ .}
 \end{aligned}$$

As an intermediary step in formulating a variational inequality, from the above equations, we can write the complementarity problem

$$\begin{aligned} u &\geq f \\ -u_{xx} + p &\geq 0 \\ (u - f)(u_{xx} - p) &= 0 \\ u(0) = u(1) &= 0 . \end{aligned}$$

We now define a Hilbert space (see Appendix A for more details on Sobolev and Hilbert spaces)

$$\mathcal{V} = \{v \in \mathcal{H}^1[0, 1] : v(0) = v(1) = 0, v \geq f\} .$$

We further define, for $u, v \in \mathcal{V}$, a bilinear form

$$a(u, v) = \int_0^1 u_x v_x dx$$

and a linear mapping

$$l(u) = - \int_0^1 p u dx .$$

Then for any $v \in \mathcal{V}$

$$\begin{aligned} a(u, v - u) &= \int_0^1 u_x (v - u)_x dx \\ &= - \int_0^1 u_{xx} (v - u) dx && \text{using integration by parts} \\ &\geq - \int_0^1 p (v - u) dx \\ &\geq l(v - u) \end{aligned}$$

which is the *variational inequality* for the problem. As stated above this could now be used to prove the existence and uniqueness of a solution using the theorem of Appendix A or for a finite element numerical solution. See Elliott and Ockendon [17] for a proof that the variational inequality implies the complementarity problem and in turn the classical problem, provided we assume the solution to the variational inequality has continuous first derivatives. Furthermore, using a projected SOR method to solve the finite element problem, which can be generated from this variational inequality, the procedure naturally determines the position of the free boundary without the need to explicitly track it.

1.6 Layout and aims

In Chapter 2 a theory is reviewed for the water entry of a uniformly nearly flat rigid convex body. The body is nearly parallel to the free surface and the effects of gravity, fluid viscosity, surface tension, air pressure and air entrapment are all neglected. This has previously been formulated and analysed by many including Wagner [100], Korobkin [47], Wilson [102] and Morgan [58] and is formulated here as a clear example of how a codimension-two free boundary problem is derived and to demonstrate the techniques and arguments that are required to address such a problem. In Section 2.2 the variational formulation of the model as first proposed by Korobkin [45] is discussed. A detailed analysis of the stability of the codimension-two free boundary, to small perturbations along its length, is undertaken. The initial value problem for small disturbances to the free surface and codimension-two free boundary are then considered. The implications of the stability analysis and initial value problem to one particular formulation of the exit problem, namely the time reversal of the codimension-two entry problem, are discussed. Some simple extensions to the model are considered in Section 2.4.

In Chapter 3 we consider three types of problems in solid mechanics. We discuss some of the concepts and methods to be used in the chapter including dynamic stress intensity factors, cohesive zones, slip, Airy stress functions and Muskhelishvili potentials. The first problem is that of a dynamic type-III crack. We begin by solving the basic problem in which the crack faces are assumed to be stress free and using a dynamic stress intensity factor determine a formula for the crack propagation speed. We then discuss the effect of using the model of Barenblatt [2, 3] and including a cohesive zone near the tip of the crack. In the next section the problem of two-dimensional contact of two identical elastic bodies is reviewed as an example of a problem whose field equation is the biharmonic equation. The problem is solved using both a superposition approach and the more elegant Muskhelishvili potential method. The results of several different loading histories are considered. The first problem solved is that of purely normal loading. The problem of a subsequent tangential loading after the initial normal loading is then considered. The effect of friction and the possibility of either no slip or some slip occurring on the contact region are discussed. The problem of simultaneous variations of the normal and tangential forces is considered in the form of incremental loadings giving rise to a sequence of static problems. The third problem of the chapter is a static type-I crack problem and shows how the Muskhelishvili potential method of the previous section is easily

modified to handle such a problem. The solution to the elliptic crack problem is then used to generate solutions to a problem in which we require the crack to close under an increasing load.

Chapter 4 concerns a problem related to the manufacture of car windscreens. The resulting codimension-two problem does not arise as a leading-order problem after exploiting a small parameter, as do all the other problems we consider in this thesis, but instead occurs naturally. The real industrial process involves placing a sheet of plate glass on a frame and heating it from above causing it to sag under its own weight. By controlling the precise heating the shape of the final windscreen is controlled. In the real problem the frame is not planar nor rectangular and the plate is sagging due to gravity and both viscous and elastic effects. We formulate a simplified version of the full problem in which we only consider a planar rectangular plate and the glass is taken to be an elastic plate. We show how the thin plate problem can be formulated as a variational inequality. In Section 4.2 the problem of a single simply-supported corner is analysed in detail to show how the edge reaction depends on the corner angle and in particular how a point force may occur at a right-angle corner. The analysis also demonstrates the possibility of the corner of the glass plate lifting off the frame. An analytical solution for a simply supported rectangle is reviewed in Section 4.3. A numerical solution of the simplified problem for different types of boundary conditions is presented. The simplified problem turned out to be one that had been considered in part already in the literature [79] and as such much of the results derived here are a review.

In Chapter 5 we move to problems in Stokes flow. The first two problems are concerned with sintering. We are mainly interested in the problem of sintering under the action of surface tension. Such a problem is hoped to give insight into the much more complicated problems that really occur when making high quality glass. Some details of the process used to make the high quality glass are discussed along with a discussion of some of the research that has already been done. The solution is complicated because unlike most of the other problems in this thesis the information is flowing out of an inner region into the codimension-two region rather than from an outer region. The matching for the problem is found to be complicated and for that reason we build up to the problem by first considering the case of zero surface tension where the flow is being driven in the outer region by sources. The use of the Riemann problem formulation is invaluable in this problem. In Section 5.2 the problem of Stokes flow with non-zero surface tension is considered. This was previously considered by Morgan [58]. In Section 5.2.7 we show how the necessary

matching information from the inner region can in this case be obtained by solving a far field problem. The analysis is then extended in the next section for more general initially local free boundary shapes. The third problem of the chapter is a model for the closure of a thin channel lying at an ice-till interface. We first solve the problem for the case of a rigid impermeable bed and then consider the coupled problem when the till is also modelled by the slow flow equations.

In Chapter 6 we consider some Hele–Shaw problems. The chapter begins with a quick review of the problem of two point injection in a Hele–Shaw cell. This is followed by the three discs problem. The problem involves a half-space of fluid moving at a constant velocity in a Hele–Shaw cell and meeting three stationary touching discs of fluid aligned along the direction of motion of the half space. This problem has been solved exactly by Richardson [76] by means of a complex variable method. We will present a codimension-two solution valid for small times after the initial impact. This problem has three codimension-two regions and the solution shows how we apply our procedure of following the flow of information when we have several codimension-two regions which are related by outer problems. In Section 6.2 this problem is generalised to the case of n discs. The following section deals with the Muskat problem [60] which is concerned with the removal of one fluid from a porous medium by injecting a second fluid to force the first out. This problem is also analogous to a two fluid Hele–Shaw problem. The problem demonstrates the extra analysis needed when the field equations in the thin region are non-trivial.

Finally conclusions are drawn in Chapter 7 and problems that remain open are listed.

1.7 Statement of originality

Originality is claimed for the particular way in which the water entry problem is solved in Section 2.1.1.1. The results of the stability analysis of Section 2.3.1 are also original work. The initial value problem of Section 2.3.2 is all new material. The numerical solution of Section 2.4.1 and the application of the non-constant velocity results to a dropped body in Section 2.4.2, for the water entry problem, are new results.

The majority of the material in Chapter 3 is review. The presentation of how contact and crack problems are codimension-two problems, however, gives them a new novel setting. The analysis of Section 3.3.1.4 is new material.

Most of Chapter 4 is a review of a problem previously looked at in 1969 [79]. It mainly serves to update the results and check their accuracy, as well as to add the correction that point forces do not occur at the lift off points and to explain when point forces may occur. However, the corner problem analysis of Section 4.2 is new work.

The Stokes flow sintering problems of Chapter 5 are original work although as discussed in Section 5.2 a previous analysis of the surface tension problem had been carried out by Morgan [58] and the inner problem solution is by Hopper [34]. The particular method used to solve the ice closure problem is new work.

The problem of sections 6.2 and an outline of it's solution were originally derived by Cummings [14]. However, the formulation as a succession of Riemann problems and the generalisation of Section 6.2.7 is all new work. The solution of the two Muskat problems of Section 6.3 is original work.

Chapter 2

The water entry problem

The water entry problem is the study of the normal impact of a nearly flat rigid convex body on an idealised fluid surface as shown below. Alternatively, the problem can be thought of as the impact of a body which has zero gradient at the initial point of contact, with the codimension-two model only being valid in a small region around the impacting body and only for small times after the initial contact. This problem was first considered by Wagner [100] in an attempt to determine the forces on a landing seaplane. It has since been considered by many others including [11, 20, 37, 46, 58, 94]. The body is assumed to move with a constant velocity and the fluid is taken to be inviscid and incompressible (since the Reynolds number $Re \sim 10^8$ and the Mach number in water $M_w \sim 10^{-2}$). For typical values of surface tension and large impact velocities of the body the Froude number and Weber number for a ship are found to be large ($Fr^2 \sim 10$, $We \sim 10^4$) which suggests that neglecting the effects of gravity and surface tension is realistic. Also the effects of air between the body and the water will be neglected and the body is assumed to be moving through a vacuum. Some of these effects are considered in [19, 27, 46, 102]. The angle that the body makes with the fluid surface is known as the deadrise angle. In the case of large

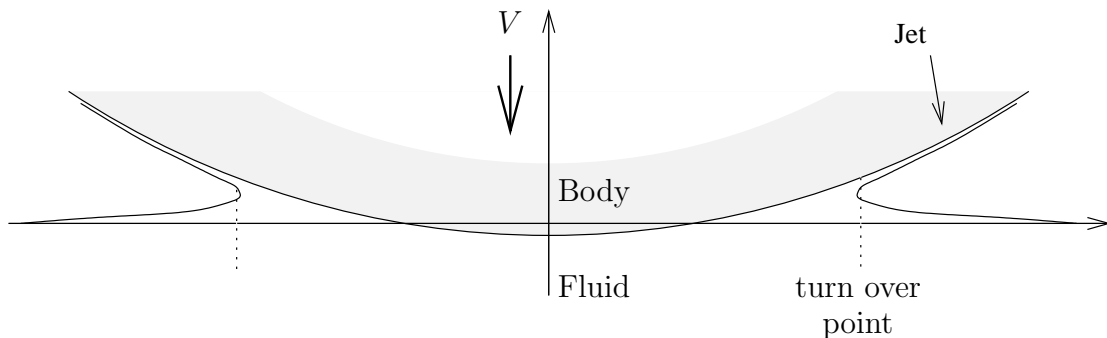


Figure 2.1: The geometry of the water entry problem.

impact velocities and small deadrise angles which we are considering the fluid surface undergoes a violent motion and there exist small regions of the flow in which large changes occur. The rapid motion of the surface and regions of large change are very important in practice and also severely hamper a numerical analysis of the problem. However both these circumstances favour an asymptotic approach.

2.1 The two dimensional water entry model

We take L to be the typical length scale over which we are interested. In dimensional variables we define the body profile by $y^* = Lf^*(\epsilon x^*/L)$, where $f^*(0) = 0$ and ϵ is a small number, and denote the velocity with which the body travels by V . Then the position of the body at a time t^* is

$$y^* = Lf\left(\frac{\epsilon x^*}{L}\right) - Vt^*.$$

We nondimensionalise by defining

$$x^* = Lx \quad , \quad y^* = Ly \quad , \quad t^* = \frac{L}{V}t$$

to obtain the nondimensional position of the body

$$y = f(\epsilon x) - t.$$

As shown in [37], for $\epsilon \ll 1$, the codimension-one formulation of the model is as shown in Figure 2.2.

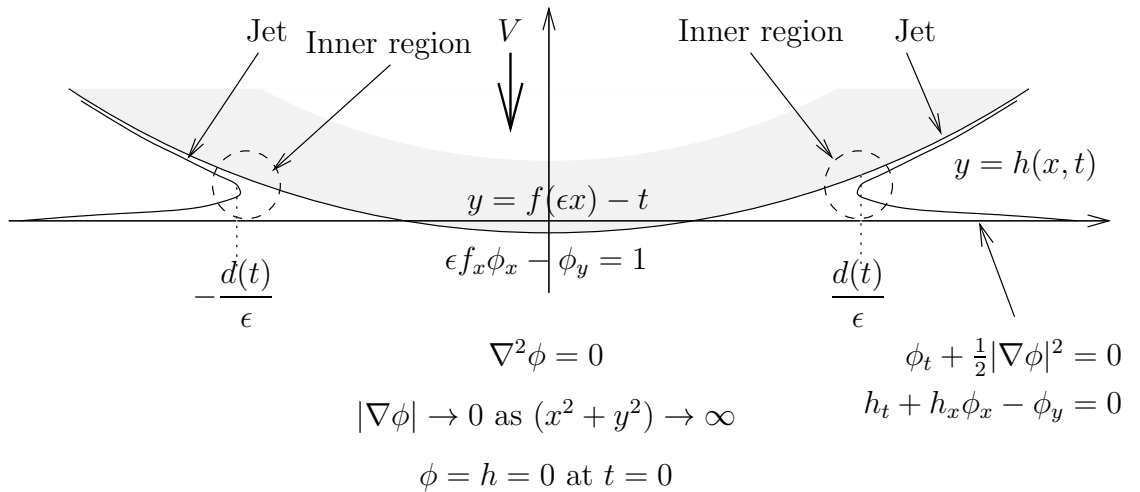


Figure 2.2: The codimension-one water entry problem.

For simplicity we have taken the free surface to be initially flat. However, the theory is easily applied to solid/liquid or liquid/solid impacts when one or both are not initially flat provided the deadrise angle is small and the initial body shape and free surface are convex or flat. The flow is initially at rest and thus initially irrotational, by Kelvin's theorem the flow is therefore always irrotational. Furthermore, as mentioned earlier we assume the flow is incompressible and hence we have a velocity potential $\phi(x, y, t)$ which satisfies Laplace's equation. On the free surface $y = h(x, t)$ we have Bernoulli and kinematic conditions. The free surface 'turns over' and forms two jets running along the body. Howison et al. [37] show that these turn-over points lie within $O(\epsilon)$ of $(\pm d(t)/\epsilon, f(\pm d(t)) - t)$. Furthermore they show that the jets only exert a second-order influence on the codimension-two model.

A natural approach now is to look for a perturbation solution by introducing certain scalings.

2.1.1 The outer problem

Relative to an $O(1)$ length scale for f the separation of the turn-over points is of $O(1/\epsilon)$. We, therefore, take the length scale of the outer problem to be $O(1/\epsilon)$. The velocity in the outer region will be $O(1)$ and thus we introduce the scaled outer variables \hat{x} , \hat{y} and $\hat{\phi}$ defined by

$$\hat{x} = \epsilon x \quad , \quad \hat{y} = \epsilon y \quad , \quad \hat{\phi} = \epsilon \phi \quad .$$

In outer coordinates the body position becomes

$$\hat{y} = \epsilon(f(\hat{x}) - t) \quad .$$

and we write the free surface $y = h(x, t)$ as

$$\hat{y} = \epsilon \hat{h}(\hat{x}, t) \quad .$$

With these scalings the free points lie at $\hat{x} = \pm d(t)$ and the jet roots lie at $(\pm d(t) + O(\epsilon^2), \epsilon(f(\pm d(t)) - t) + O(\epsilon^2))$. It is now reasonable to linearise the boundary conditions onto the x axis and to ignore the jets (this assumption is shown to be valid by Howison et al. [37]) so that

$$\hat{h}(d(t), t) = f(d(t)) - t$$

to leading order.

Bibliography

- [1] J.M. AITCHISON. Percolation in gently sloping beaches. *IMA J. Appl. Math.*, **33**:17–31, 1984.
- [2] G.I. BARENBLATT. Concerning equilibrium cracks forming during brittle fracture: The stability of isolated cracks, relationship with energetic theories. *PMM Applied Mathematics and Mechanics*, **23**:1273–1282, 1959.
- [3] G.I. BARENBLATT. The formation of equilibrium cracks during brittle fracture: General ideas and hypotheses, axially symmetric cracks. *PMM Applied Mathematics and Mechanics*, **23**:622–636, 1959.
- [4] G.I. BARENBLATT. *Scaling, self-similarity, and intermediate asymptotics*. Cambridge texts in applied mathematics. Cambridge University Press, 1996.
- [5] G.I. BARENBLATT, R.L. SALGANIK, AND G.P. CHEREPANOV. On the non-steady motion of cracks. *PMM Applied Mathematics and Mechanics*, **26**:469–477, 1962.
- [6] G.K. BATCHELOR. *An Introduction to Fluid Mechanics*. Cambridge University Press, 1967.
- [7] M.L. BLANPIED AND T.E. TULLIS. The stability of a frictional system with a two state variable constitutive law. *Pageoph*, **124**:413–433, 1986.
- [8] A. BOSSAVIT, A. DAMLAMIAN, AND M FRÉMOND, editors. *Free Boundary Problems: Applications and Theory*, *Pitman Research Notes Math.*, **120,121**, London, 1985. Pitman.
- [9] S.L. CECCIO AND C.E. BRENNEN. Observations of the dynamics and acoustics of travelling bubble cavitation. *J. Fluid Mech.*, **233**:633–660, 1991.

- [10] J.M. CHADAM AND M. RASMUSSEN, editors. *Emerging Applications in Free Boundary Problems, Pitman Research Notes Math.*, **280,281**. Longman, Harlow, 1993.
- [11] R. COINTE AND J.L. ARMAND. Hydrodynamic impact analysis of a cylinder. *J. Off. Mech. Arc. Eng.*, **107**:237–243, 1987.
- [12] J. CRANK. *Free and Moving Boundary Problems*. Clarendon Press, Oxford, 1984.
- [13] C.W. CRYER. A bibliography of free boundary problems, technical summary report. Technical report, Math. Research Centre, N. 1793 Wisconsin, 1977.
- [14] L. CUMMINGS. The three cylinder problem in Hele–Shaw flow. private communication, 1997.
- [15] J.H. DIETERICH. Modeling of rock friction, 1. Experimental results and constitutive equations. *J. Geophys. Res.*, **84**:2161–2168, 1979.
- [16] D.C. DUGDALE. Yielding of steel sheets containing slits. *J. Mech. Phys. Solids*, **8**:100–104, 1960.
- [17] C.M. ELLIOTT AND J.R. OCKENDON. *Weak and Variational Methods*, **59**. Pitman Research Notes, London, 1982.
- [18] A. FASANO AND M. PRIMERCERO, editors. *Free Boundary Problems: Theory and Applications, Pitman Research Notes Math.*, **78,79**. Pitman, London, 1983.
- [19] E. FONTAINE AND R. COINTE. A slender body approach to nonlinear bow waves. *Phil. Trans. Roy. Soc. Lon. A*, **355**:565–574, 1997.
- [20] L. E. FRAENKEL AND J.B. MCLEOD. Some results for the entry of a blunt wedge into water. *Phil. Trans. Roy. Soc. Lon. A*, **355**:523–535, 1997.
- [21] L.E. FRAENKEL. On the method of matched asymptotic expansions (parts I–III). In *Proceedings of the Cambridge Philosophical Society*, **65**, pages 209–284, 1969.
- [22] L.B. FREUND. *Dynamic Fracture Mechanics*. CUP, 1990.

- [23] A. FRIEDMAN. *Variational Principles and Free Boundary Problems*. J. Wiley, New York, 1982.
- [24] G.F. GAKHOV. *Boundary Value Problems*, **85** of *International series of monographs in pure and applied mathematics*. Pergamon, 1966.
- [25] J.W. GLASHEEN AND T.A. MCMAHON. A hydrodynamic model of locomotion in the basilisk lizard. *Nature*, **380**:340–342, 1996.
- [26] A.E. GREEN AND W. ZERNA. *Theoretical Elasticity*. Dover, 19668.
- [27] M. GREENHOW AND S. MAYO. Water entry and exit of horizontal circular cylinders. *Phil. Trans. Roy. Soc. Lon. A*, **355**:551–563, 1997.
- [28] A.A. GRIFFITH. The phenomenon of rupture and flow in solids. *Phil. Trans. Roy. Soc. Lon. A*, **221**:163–198, 1920.
- [29] H. HERTZ. Über die berührung fester elastischer körper (On the contact of elastic solids). *J. reine und angewandte Mathematik*, **92**:156–171, 1882.
- [30] H. HERTZ. Über die berührung fester elastischer körper und über die harte (On the contact of elastic solids and on hardness). *Verhandlungen des Vereins zur Beförderung des Gewerbefleisses, Leipzig, Nov*, 1882.
- [31] K.-H. HOFFMAN AND J. SPREKELS, editors. *Free Boundary Problems: Theory and Applications*, *Pitman Research Notes Math.*, **185,186**. Pitman, London, 1990.
- [32] R.W. HOPPER. Plane Stokes flow driven by capillarity on a free surface. *J. Fluid Mech.*, **213**:349–375, 1990.
- [33] R.W. HOPPER. Stokes flow of a cylinder and half-space driven by capillarity. *J. Fluid Mech.*, **243**:171–181, 1992.
- [34] R.W. HOPPER. Capillary-driven plane Stokes flow exterior to a parabola. *Q. J. Mech. Appl. Math.*, **46**:193–210, 1993.
- [35] S.D. HOWISON. Codimension-two free boundary problems. In J. CHADAM AND H. RASMUSSEN, editors, *Proc. Int'l Colloquium on Free Boundary Problem*. Pitman, London, 1991.

- [36] S.D. HOWISON, J.D. MORGAN, AND J.R. OCKENDON. Codimension-two free boundary problems. *SIAM Review*, **39**(2):221–253, 1997.
- [37] S.D. HOWISON, J.R. OCKENDON, AND S.K. WILSON. Incompressible water-entry problems at small deadrise angle. *J. Fluid Mech.*, **222**:215–230, 1991.
- [38] S.D. HOWISON AND S. RICHARDSON. Cusp development in free boundaries, and two-dimensional slow viscous flows. *Euro. J. Appl. Math.*, **6**:441–454, 1995.
- [39] B.J. HUNTON. *Vortex Dynamics*. DPhil thesis, Oxford, 1994.
- [40] G.R. IRWIN. Analysis of stresses and strains near the end of a crack traversing a plate. *J. Appl. Mech.*, **24**:361–364, 1957.
- [41] G.P. IVANTSOV (Г. П. Иванцов). The temperature field around a spherical, cylindrical, or point crystal growing in a cooling solution (Температурное поле вокруг шарообразного, цилиндрического и иглообразного кристалла, растущего в переохлажденном расплаве). *Doklady Akademii Nauk SSSR (Доклады Академии Наук СССР)*, **58**:567–569, 1947. (In Russian).
- [42] A. JAGOTA AND P.R. DAWSON. Simulation of the viscous sintering of particles. *J. Am. Ceramic Soc.*, **73**:173–77, 1990.
- [43] K.L. JOHNSON. *Contact Mechanics*. Cambridge University Press, 1985.
- [44] J.R. KING, H. OCKENDON, AND J.R. OCKENDON. The Laplace–Young equation near a corner. Submitted to *Quart. J. Mech. Appl. Math.*, 1998.
- [45] A.A. KOROBKIN. Formulation of penetration problem as a variational inequality. *Din. Sploshnoi Sredy*, **58**:73–79, 1982.
- [46] A.A. KOROBKIN. Asymptotic theory of liquid-solid impact. *Phil. Trans. Roy. Soc. Lon. A*, **355**:507–522, 1997.
- [47] A.A. KOROBKIN AND V.V. PUKHNACHOV. Initial stages of water impact. *Ann. Rev. Fluid Mech.*, **20**:159–185, 1988.
- [48] D.D. KOSLOFF AND H.P. LIU. Reformulation and discussion of mechanical behaviour of the velocity dependent friction law proposed by Dieterich. *Geophys. Res. Lett.*, **7**:913–916, 1980.

- [49] H.K. KUIKEN. Viscous sintering: the surface-tension-driven flow of a liquid form under the influence of curvature gradients at its surface. *J. Fluid Mech.*, **214**:503–515, 1990.
- [50] M.K. KUO. Transient stress intensity factors for a cracked plane strip under anti-plane point forces. *Int. J. Eng. Sci.*, **30**:199–211, 1992.
- [51] A.A. LACEY, S.D. HOWISON, J.R. OCKENDON, AND P. WILMOTT. Irregular morphologies in unstable Hele-Shaw free-boundary problems. *Quart. J. Mech. Appl. Math.*, **43**:387–405, 1990.
- [52] A.A. LACEY AND A.B. TAYLER. A mushy region in a Stefan problem. *IMA J. Appl. Math.*, **30**:303–313, 1983.
- [53] H.W. LIEPMANN AND A. ROSKO. *Elements of Gas Dynamics*. Wiley, 1957.
- [54] M.F. LINKER AND J.H. DIETERICH. Effects of variable normal stress on rock friction: Observations and constitutive equations. *J. Geophys. Res.*, **97**:4923–4940, 1992.
- [55] J.I. MARTINEZ-HERRERA AND J.J. DERBY. Analysis of capillary-driven viscous flows during the sintering of ceramic powders. *AIChE Journal*, **40**:1794–1803, 1994.
- [56] R.D. MINDLIN AND H. DERESIEWICZ. Elastic spheres in contact under varying oblique forces. *J. Appl. Mech.*, **20**:327–344, 1953.
- [57] M. MOGHISI AND P.T. SQUIRE. An experimental investigation of the initial force of impact on a sphere striking a liquid surface. *J. Fluid Mech.*, **108**:133–146, 1981.
- [58] J.D. MORGAN. *Codimension-two free boundary problems*. DPhil thesis, Oxford, 1994.
- [59] J.D. MORGAN, D.L. TURCOTTE, AND J.R. OCKENDON. Models for earthquake rupture propagation. *Tectonophysics*, **277**:209–217, 1997.
- [60] M. MUSKAT. Two fluid systems in porous media. The encroachment of water into an oil sand. *Physics*, **5**:250–264, 1934.
- [61] N.I. MUSKHELISHVILI. *Some Basic Problems of the Mathematical Theory of Elasticity*. P. Noordhoff Ltd, 1953. (English translation by J.R.M. Radok).

- [62] H. NAKANISHI. Continuum model of mode-III crack propagation with surface friction. *Physics Review E*, **49**:5412–5419, 1994.
- [63] F. NG. *Mathematical Modelling of Subglacial Drainage and Erosion*. DPhil thesis, Oxford, 1998.
- [64] M. NIEZGODKA AND I. PAVLOW. Recent advances in free boundary problems. *Control Cybernetics*, **14**:1–307, 1985.
- [65] F. NILSSON. Dynamic stress-intensity factors for finite strip problems. *Int. J. Fract. Mech.*, **8**:403–411, 1972.
- [66] D. NOWELL, D.A. HILLS, AND A. SACKFIELD. Contact of dissimilar elastic cylinders under normal and tangential loading. *J. Mech. Phys. Solids*, **36**:59–75, 1988.
- [67] J.R. OCKENDON. A class of moving boundary problems arising in industry. In *Proc. Venice Conf. Appl. Ind. Math.* Kluwer Academic Publishers, 1991.
- [68] J.R. OCKENDON AND W.R. HODGKINS. *Moving Boundary Problems in Heat Flow and Diffusion*. Clarendon Press, Oxford, 1975.
- [69] P.G. OKUBO. Dynamic rupture modeling with laboratory-derived constitutive relations. *J. Geophys. Res.*, **94**:12321–12335, 1989.
- [70] K. O’MALLEY, A.D. FITT, T.V. JONES, J.R. OCKENDON, AND P. WILMOTT. Models for high Reynolds number flow down a step. *J. Fluid Mech.*, **222**:139–155, 1991.
- [71] A. PROSPERETTI AND H.N. OGUZ. Surface-tension effects in the contact of liquid surfaces. *J. Fluid Mech.*, **203**:149–171, 1989.
- [72] J.N. REDDY. *Applied Functional Analysis and Variational Methods in Engineering*. Keiger publishing company, 1991.
- [73] R.L. RICCA. Rediscovery of da Rios equations. *Nature*, **352**:561–562, 1992.
- [74] S. RICHARDSON. Some Hele-Shaw flows with time-dependent free boundaries. *J. Fluid Mech.*, **102**:263–278, 1981.
- [75] S. RICHARDSON. Two-dimensional slow viscous flows with time-dependent free boundaries driven by surface tension. *Euro. J. Appl. Math.*, **3**:193–207, 1992.

- [76] S. RICHARDSON. Hele–Shaw flows with free boundaries driven along infinite strips by a pressure difference. *Euro. J. Appl. Math.*, **7**:345–366, 1996.
- [77] J.C.W. ROGERS AND W.G. SZYMCAK. Computation of violent surface motion: comparison with theory and experiment. *Phil. Trans. Roy. Soc. Lon. A*, **355**:649–664, 1997.
- [78] K.R. RUSHTON. Dynamic-relaxation solutions of elastic-plate problems. *J. Strain Anal.*, **3**:23–32, 1968.
- [79] K.R. RUSHTON. Simply supported plates with corners free to lift. *J. Strain Anal.*, **4**:307–311, 1969.
- [80] P.G. SAFFMAN. *Vortex Dynamics*. CUP, 1992.
- [81] N. SHIMAKURA. *Partial Differential Operators of Elliptic Type*. American Mathematical Society, 1992.
- [82] G.C. SIH, editor. *Methods of analysis and solution of crack problems*. Noordhoff Int., 1973.
- [83] G.C. SIH AND E.P. CHEN. Moving cracks in a finite strip under tearing action. *J. Franklin Inst.*, **290**:25–35, 1970.
- [84] I.N. SNEDDON AND D.S. BERRY. *The classical theory of elasticity*. Springer–Verlag, 1958.
- [85] I.N. SNEDDON AND R. HILL, editors. *Progress in Solid Mechanics*. North-Holland, 1963.
- [86] I.N. SNEDDON AND M. LOWENGRUB. *Crack Problems in the classical theory of elasticity*. Wiley, 1969.
- [87] I.S. SOKOLNIKOFF. *Mathematical theory of elasticity*. McGraw–Hill, 1956.
- [88] D.A. SPENCE. Self similar solutions to adhesive contact problems with incremental loading. *Proc. Roy. Soc.*, **305A**:55–80, 1968.
- [89] D.A. SPENCE. The Hertz contact problem with finite friction. *J. Elasticity*, **5**:297–319, 1975.
- [90] D.A. SPENCE. Frictional contact with transverse shear. *Quart. J. Mech. Appl. Math.*, **39**:233–253, 1986.

- [91] D.A. TARZIA. A bibliography of moving-free boundary problems for the heat diffusion equation, 1988.
- [92] S. TIMOSHENKO AND S. WOINOWSKI-KREIGER. *Theory of Plates and Shells*. McGraw-Hill Book Co. Inc., 2nd edition, 1959.
- [93] S.P. TIMOSHENKO AND J.N. GOODIER. *Theory of elasticity*. McGraw-Hill, 3rd edition, 1970.
- [94] W. TOLLMIEH. Zum landestoß von seeflugzeugen. *Zeitschrift Für Angewandte Mathematik Und Mechanik*, **14**:251, 1934.
- [95] J.R. TURNER. The frictional unloading problem on linear elastic half-space. *J. Inst. Maths Applics*, **24**:439–469, 1979.
- [96] G.A.L. VAN DE VORST. Integral method for a two-dimensional Stokes flow with shrinking holes applied to viscous sintering. *J. Fluid Mech.*, **257**:667–689, 1993.
- [97] G.A.L. VAN DE VORST. Numerical simulation of axisymmetric viscous sintering. *Eng. Anal. with Boundary Elements*, **14**:193–207, 1994.
- [98] G.A.L. VAN DE VORST, R.M.M. MATTHEIJ, AND H.K. KUIKEN. A boundary element solution for two-dimensional viscous sintering. *J. Comp. Phys.*, **100**:50–63, 1992.
- [99] M. VAN DYKE. *Perturbation methods in fluid dynamics*. Parabolic Press, annotated edition, 1975.
- [100] H. WAGNER. Über stoß- und gleitvorgänge an der oberfläche von flüssigkeiten (phenomena associated with impacts and sliding on liquid surfaces). *Zeitschrift Für Angewandte Mathematik Und Mechanik*, **12**:193–215, 1932.
- [101] D.G. WILSON, A.D. SOLOMON, AND P.T. ROGGS, editors. *Moving Boundary Problems*. Academic Press, New York, 1978.
- [102] S.K. WILSON. *The Mathematics of ship slamming*. DPhil thesis, Oxford, 1989.