DISCONNECTEDNESS AND COMPACTNESS-LIKE PROPERTIES IN HYPERSPACES

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1. **Notations and basic definitions**

Every space in this talk will be a Tychonoff space.

For a topological space X, let $\mathcal{CL}(X)$ be the hyperspace of nonempty closed subsets of X with the Vietoris Topology.

Let us consider the following subspaces of $\mathcal{CL}(X)$:

$$\mathcal{K}(X) = \{K \in \mathcal{CL}(X) : K \text{ is compact}\},$$

$$\mathcal{F}_n(X) = \{F \in \mathcal{CL}(X) : |F| \leq n\},$$

and

$$\mathcal{F}(X) = \bigcup_{n \in \mathbb{N}} F_n(X).$$

Recall that the Vietoris topology in $\mathcal{CL}(X)$ has the collection of all the sets of the form

$$V^{+} = \{A \in \mathcal{CL}(X) : A \subseteq V\}$$

and

$$V^{-} = \{A \in \mathcal{CL}(X) : A \cap V \neq \emptyset\}$$

where V is an open subset of X, as a subbase.

So, given open subsets U_1,...,U_n of X, the set

$$<U_1, \ldots, U_n> = \{T \in \mathcal{CL}(X) : T \in (\cup_{1 \leq k \leq n} U_k)^{+}$$
$$\text{and } \; T \in U_k^{-} \text{ for each } 1 \leq k \leq n\},$$

is a canonical open set in $\mathcal{CL}(X)$.

I am going to talk to you about some results that Juan Angoa, Rodrigo Hernández-Gutierrez, Yasser Ortiz-Castillo and I obtained about compactness-like and disconnectness-like properties of hyperspaces with their Vietoris Topology.

2. **Properties related to compactness in hyperspaces**

In 1951, E.A. Michael proved:

Theorem 2.1. *A space X is compact if and only if $\mathcal{CL}(X)$ is compact;*

With respect to countable compactness, in 1985 D. Milovaněvić proved:

Theorem 2.2. *The following statements are equivalent:*

(1) *Every σ-compact set of X has a compact closure in X (X is ω-hyperbounded);*
(2) *$\mathcal{K}(X)$ is countably compact;*
(3) *$\mathcal{K}(X)$ is ω-bounded; and*
(4) *Every σ-compact subspace of $\mathcal{K}(X)$ has a compact closure in $\mathcal{K}(X)$ ($\mathcal{K}(X)$ is ω-hyperbounded).*

Recall that a space X is ω-bounded if every countable subset of X is contained in a compact subspace of X.

With respect to Milovaněvić's result we obtained the following theorem. First, some definitions.

Definitions 2.3. Let X be a topological space.

(1) X is *initially* κ*-compact* if every set $A \in [X]^{\leq\kappa}$ has a complete accumulation point.

(2) X is κ*-bounded* if every set $A \in [X]^{\leq\kappa}$ has a compact closure in X;

(3) X is κ*-hyperbounded* if for each family $\{S_\xi : \xi < \kappa\}$ of compact sets of X, $Cl_X(\cup_{\xi<\kappa} S_\xi)$ is a compact subspace.

Theorem 2.4. *(J. Angoa, Y.F. Ortiz-Castillo and Á. Tamariz-Mascarúa, 2011)*

Let X be an space and let κ be an infinite cardinal. Then the following statements are equivalent:

(1) X *is* κ*-hyperbounded;*

(2) $\mathcal{K}(X)$ *is initially* κ*-compact;*

(3) $\mathcal{K}(X)$ *is* κ*-bounded; and*

(4) $\mathcal{K}(X)$ *is* κ*-hyperbounded.*

For example, for every infinite cardinal κ, $\mathcal{K}([0, \kappa^+))$ is initially κ-compact. Moreover, $\mathcal{K}(\Sigma(\{0,1\}^{\omega_1}, \mathbf{0}))$ is not countably compact.

3. **Pseudocompactness of hyperspaces**

We obtained a characterization of pseudocompactness of $\mathcal{K}(X)$ in terms of the following property in X:

Definition 3.1. Let X be a space. We say that X is pseudo-ω-bounded if for each countable family $\mathcal{U}$ of non-empty open subsets of X, there exists a compact set $K \subseteq X$ such that, for each $U \in \mathcal{U}$, $K \cap U \neq \emptyset$.

Theorem 3.2. (J. Angoa, Y.F. Ortiz-Castillo, Á. Tamariz-Mascarúa, 2011)

Let X be a space. Then the following statements are equivalent:

(1) X is pseudo-ω-bounded;

(2) $\mathcal{K}(X)$ is pseudocompact; and

(3) $\mathcal{K}(X)$ is pseudo-ω-bounded.

For example, for every $p \in \beta\omega \setminus \omega$, $\mathcal{K}(\beta\omega \setminus \{p\})$ is not pseudocompact, and $\mathcal{K}(\Sigma(\{0,1\}^{\omega_1}, \mathbf{0}))$ is pseudocompact.

When $\mathcal{K}(X)$ is C^*-embedded in $\mathcal{CL}(X)$, we obtained:

Theorem 3.3. *(*J. Angoa, R. Hernández-Gutiérrez, Y.F. Ortiz-Castillo and A. Tamariz-Mascarúa*, 2011)*
Let X be a space such that $\mathcal{K}(X)$ is C^*-embedded in $\mathcal{CL}(X)$. Then the next statements are equivalent:

(1) *X is compact,*
(2) *X is σ-compact,*
(3) *$\mathcal{K}(X)$ is compact,*
(4) *$\mathcal{K}(X)$ is σ-compact,*
(5) *$\mathcal{K}(X)$ is Lindelöf, and*
(6) *$\mathcal{K}(X)$ is paracompact.*

4. **Disconnectness of hyperspaces and some related topics**

Recall that a space X is

(1) *zero-dimensional* if each point in X has a local base of neighborhoods constituted by clopen subsets of X;

(2) *totally disconnected* if for every pair of points $x, y \in X$ with $x \neq y$, there is a clopen set O such that $x \in O$, $y \notin O$; and

(3) *hereditarily disconnected* if the only non-empty connected subspaces of X are those having only one point.

Of course, every zero-dimensional space is totally disconnected and the totally disconnected spaces are herditarily disconnected.

Theorem 4.1. *(E.A. Michael, 1951) For a space X we have that:*

(1) *X is connected if and only if $\mathcal{H}$ is connected where $\mathcal{F}(X) \subseteq \mathcal{H} \subseteq \mathcal{CL}(X)$.*

(2) *X is discrete if and only if $\mathcal{K}(X)$ is discrete,*

(3) *X is zero-dimensional if and only if $\mathcal{K}(X)$ is zero-dimensional,*

(4) *X is totally disconnected if and only if $\mathcal{K}(X)$ is totally disconnected.*

Now, we present some results about classes of highly disconnected spaces.

(1) If X is a space and $p \in X$, we call p a *P-point* of X if p belongs to the interior of every G_δ set that contains it.

(2) We say that X is a *P-space* if all its points are P-points of X.

(3) A *basically disconnected* space is a space in which every cozero set has open closure.

(4) A space is *extremely disconnected* if every open set has open closure.

Definition 4.2. (1) An F*-space* is a space in which every cozero set is C^*-embedded.

(2) We may also consider F'-spaces, that is, spaces in which each pair of disjoint cozero sets have disjoint closures.

Proposition 4.3. *(R.J. Hernández-Gutiérrez and Á. Tamariz-Mascarúa, 2010)*
Let X be a space and $\mathcal{F}_2(X) \subseteq \mathcal{H} \subseteq \mathcal{K}(X)$. Then $\mathcal{H}$ is extremely disconnected if and only if X is discrete.

Theorem 4.4. *(R.J. Hernández-Gutiérrez and Á. Tamariz-Mascarúa, 2010)*
Let X be a space and $\mathcal{F}_2(X) \subseteq \mathcal{H} \subseteq \mathcal{K}(X)$. Then the following are equivalent:

(a) *X is a P-space,*
(b) *$\mathcal{H}$ is a P-space,*
(c) *$\mathcal{H}$ is basically disconnected,*
(d) *$\mathcal{H}$ is an F-space, and*
(e) *$\mathcal{H}$ is an F'-space.*

So, $\beta\omega$ is basically disconnected but $\mathcal{K}(\beta\omega)$ is not basically disconnected, and $\mathcal{K}(\Box_\omega\{0,1\}^{\omega_1})$ is a P-space.

The most interesting question about disconnectedness of hyperspaces is:

When is $\mathcal{CL}(X)$ or $\mathcal{K}(X)$ hereditarily disconnected?

Problem 4.5. (A. Illanes y S. Nadler, 1999) *Is either $\mathcal{CL}(X)$ or $\mathcal{K}(X)$ hereditarily disconnected when X is metrizable and hereditarily disconnected?*

E. Pol and R. Pol proved in 2000 that the answer to this question is in the negative by giving some examples. Afterwards, we obtained the following result which gives a method to locate connected sets in a hyperspace.

Proposition 4.6. *(R.J. Hernández-Gutiérrez and Á. Tamariz-Mascarúa, 2010) Let X be a space. Assume there is $K \in \mathcal{K}(X)$ such that X is the only clopen subset of X containing K. Then $\mathcal{C} = \{K \cup \{x\} : x \in X\}$ is a connected subspace of $\mathcal{K}(X)$.*

For example, the Knaster-Kuratowski fan without its vertex, $\mathbb{F}$, is hereditarily disconnected but $\mathcal{K}(\mathbb{F})$ is not hereditarily disconnected. In fact there is a function h from the Cantor set C to $[0, \frac{1}{2})$ such that the graph of h is a compact subset K of $\mathbb{F}$ satisfying the conditions in Proposition 4.5.

The Main Theorem of this talk is the following:

Theorem 4.7. *(R.J. Hernández-Gutiérrez and Á. Tamariz-Mascarúa, 2010)*
Assume that there is a closed subset F of X such that

(a) *F and $X \setminus F$ are totally disconnected, and*

(b) *the quotient space X/F is hereditarily disconnected.*

Then, $\mathcal{K}(X)$ is hereditarily disconnected.

Moreover, we obtained a partial convers:

Theorem 4.8. *(R.J. Hernández-Gutiérrez and Á. Tamariz-Mascarúa, 2010)*
Assume that $X = Y \cup T$ where Y and T are totally disconnected and T is compact. Then, $\mathcal{K}(X)$ is hereditarily disconnected if and only if X/T is hereditarily disconnected.

As an application of these theorems we obtained the following examples.

Let $\phi : \{0,1\}^\omega \to [0,\infty]$ be the function defined as

$$\phi(t) = \Sigma_{n<\omega}\frac{t_m}{m+1} \text{ where } t = (t_n)_{n<\omega}.$$

Take $X = \{x \in \{0,1\}^\omega : \phi(x) < \infty\}$,

$X_0 = \{(x, \phi(x)) : x \in X\}$, and

let $Y = X_0 \cup (X \times \{\infty\})$.

It happens that $\mathcal{K}(Y)$ is hereditarily disconnected.

On the other hand, if we take

$$Z = X_0 \cup (2^\omega \times \{\infty\},$$

it can be proved that $\mathcal{K}(Y)$ is not hereditarily disconnected.

5. **Problem**

Problem 5.1. *What we can say about beeing* $\mathcal{K}(X)$ *(or* $\mathcal{CL}(X)$*) hereditarily disconnected when we have a richer structure in* X*; for example, when* X *is a topological group?*