Betweenness in a Continuum: Lessons from the Crooked Torus

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15th Galway Topology Colloquium,
Mathematical Institute, University of
Oxford, 09–11 July, 2012

1. Betweenness via Road Systems

2. Subcontinuum Road Systems

3. A Characterization Problem

4. The Crooked Torus

5. Proof Outline for 3.1

6. Summary

1. Betweenness via Road Systems. We take the intuitive view that point c lies between points a and b exactly when every "road" allowing travel from a to b (and *vice versa*) must go through c.

This "roadblock" vision of betweenness has led to the following simple abstract definition:

- A *road system* is a pair $\langle X, \mathcal{R} \rangle$, where X is a nonempty set and $\mathcal{R}$ is a family of subsets of X, called *roads*, satisfying:
  - Every singleton subset of X is a road.
  - Every doubleton subset of X is contained in at least one road.
  - (Additivity Condition): The union of two intersecting roads is a road.

If $\langle X, \mathcal{R} \rangle$ is a road system and $a, b \in X$, the set of points c *between* a and b is denoted $[a, b]$ and is the set $\bigcap\{R \in \mathcal{R} : a, b \in R\}$.

The *interval membership relation* $c \in [a, b]$ defines a ternary relation on the underlying set X.

A natural question is whether one may characterize—using first-order terms involving an abstract ternary relation symbol—exactly when a ternary relation $B \subseteq X^3$ is the interval membership relation arising from a road system on X.

This question has an affirmative answer.

1.1 Theorem (Road Representation): *Let B be a ternary relation on a nonempty set X. Then there is a road system $\mathcal{R}$ on X with interval membership relation B iff B satisfies the following five first-order conditions:*

R1 (Symmetry) $B(a,c,b) \to B(b,c,a)$.

R2 (Reflexivity) $B(a,b,b)$.

R3 (Minimality) $B(a,c,a) \to c = a$.

R4 (Convexity) $(B(a,c,b) \wedge B(a,d,b) \wedge B(c,e,d)) \to B(a,e,b)$.

R5 (Disjunctivity) $B(a,x,b) \to (B(a,x,c) \vee B(c,x,b))$.

2. Subcontinuum Road Systems. There are many natural situations, especially in the theory of trees and in topology, where road systems come up; the one I want to discuss today concerns roads that consist of the subcontinua of a continuum (= connected compact Hausdorff space).

In this setting $c \in [a, b]$ means that there is no subcontinuum of $X \setminus \{c\}$ that also contains $\{a, b\}$. (In particular, a point that lies between two other points in a continuum is a weak cut point of the continuum. Moreover, if X is *aposyndetic*–i.e., two points may be separated by a subcontinuum that contains one of them in its interior and misses the other–then c is actually a cut point.)

Intervals in continua are generally closed; when they're also subcontinua, we call the continuum *interval connected*.

For example, arcs are interval connected, as are dendrites in general. The $\sin(\frac{1}{x})$-continuum is another example. At the opposite extreme, in a simple closed curve any interval $[a, b]$ consists of the bracketing points alone. Such intervals, when $a \neq b$, are called *gaps*.

Recall that a continuum is *hereditarily unicoherent* if the intersection of any two of its overlapping subcontinua is a subcontinuum.

2.1 Proposition: *A continuum is interval connected iff it is hereditarily unicoherent.*

3. A Characterization Problem. The issue we wish to focus on today concerns the question of characterizing—in first-order betweenness terms—the property of being interval connected.

This question is not yet answered, but here are some plausible characterization sentences, listed in order of nondecreasing logical strength.

(Gap-free Property):
$\forall a \forall b[a \neq b \to \exists c(c \in [a,b] \wedge c \neq a \wedge c \neq b)]$

(Gap-filling Property):
$\forall a \forall b[a \neq b \to \exists c(c \in [a,b] \wedge c \neq a \wedge b \notin [a,c])]$

(Composite Property):
$\forall a \forall b[a \neq b \to \exists c(c \in [a,b] \wedge a \notin [c,b] \wedge b \notin [a,c])]$

The gap-free property clearly follows from interval connectedness; and, using a simple "boundary bumping" argument, we can show that the gap-filling property does as well. Not so the composite property.

3.1 Theorem: *A continuum satisfies the composite property iff each of its nondegenerate intervals is a decomposable subcontinuum.*

And when we strengthen gap-freeness in a completely different way, we get an even stronger condition on intervals. To explain this, first define a continuum (or any road system) to be *antisymmetric* if $[a, b] = [a, c]$ implies $b = c$. This is clearly a first-order property, it's present in aposyndetic continua, and we have:

3.2 Theorem: *A continuum is antisymmetric and satisfies the gap-free property iff each of its nondegenerate intervals is a generalized arc.*

4. The Crooked Torus. A continuum is *hereditarily indecomposable* if the intersection of any two of its overlapping subcontinua is one or the other of them. The celebrated pseudo-arc is an example of this phenomenon.

The composite property is too strong to characterize interval connectedness in general because hereditarily indecomposable continua are hereditarily unicoherent; hence intervals are indecomposable subcontinua.

But the ever so slightly weaker gap-filling property is *too* weak.

Define a continuum X to be a *crooked torus* if it may be decomposed as a union $K \cup M$ of two hereditarily indecomposable subcontinua such that $K \cap M$ has exactly two components, each nondegenerate.

4.1 Theorem: *Every crooked torus satisfies the gap-filling property, while failing to be interval connected.*

Some remarks: Let $X = K \cup M$, where K, M are subcontinua such that $K \cap M$ is a union $A \cup B$ of disjoint nondegenerate subcontinua.

(1) If $a \in A$ and $b \in B$, then $[a, b]$ is clearly not connected.

(2) If H is a subcontinuum of X that intersects both K and M, and if C is a component of H in K, then C intersects M. ("Boundary bumping," just uses fact that $X = K \cup M$.)

Now assume that both K and M are hereditarily indecomposable.

(3) If H is a subcontinuum of X that intersects both A and B, then $A \cup B \subseteq H$.

(4) Hence, if $a \in A$ and $b \in B$, then $[a, b] \supseteq A \cup B$. (In fact, they're equal.)

(5) In general, we show X satisfies gap filling by proving that, no matter where a, b lie in X, $[a, b]$ is either connected, or contains two nondegenerate disjoint subcontinua, one containing a, the other containing b.

(6) A crooked torus also satisfies another consequence of being interval connected, namely the *centroid property*: for any $a, b, c \in X$, $[a, b] \cap [a, c] \cap [b, c] \neq \emptyset$.

5. Proof Outline for 3.1.

(1) If $[a, b]$ decomposes into $K \cup M$, both proper subcontinua, then any $c \in K \cap M$ witnesses that the composite property holds.

(2) If the composite property holds and intervals are connected, then the nondegenerate ones are easily seen to be decomposable.

(3) If A and B are disjoint nonempty closed subsets of X, a Zorn's lemma argument allows you to find $a \in A$ and $b \in B$ such that for any $a' \in A$, $b' \in B$, if $[a', b'] \subseteq [a, b]$, then $[a', b'] = [a, b]$. (a and b are *minimally close*).

(4) In the absence of interval connectedness, we have subcontinua K, M with $K \cap M = A \cup B$, where A and B are closed, nonempty, and disjoint. Let $a \in A$ and $b \in B$ be minimally close (relative to A, B). If $c \in [a, b]$, then either $c \in A$ or $c \in B$. In the first case $[c, b] = [a, b]$; in the second $[a, c] = [a, b]$. Thus the composite property fails for X.

6. Summary.

Call a property $\mathfrak{P}$ of continua *B-definable* if there is a first-order sentence ϕ in an alphabet with equality and one ternary predicate symbol, such that a continuum is in class $\mathfrak{P}$ iff the corresponding interval membership relation satisfies ϕ.

Examples of properties that are B-definable include:

- Having every nondegenerate interval a decomposable continuum.
- Having every nondegenerate interval a generalized arc.
- Being hereditarily indecomposable.
- Being irreducible.

Examples of properties that are *not* B-definable include:

- Being of dimension n.
- Being chainable.
- Being homogeneous.
- Being self-similar.

And in addition to our focus question of whether being interval connected (= hereditarily unicoherent) is B-definable, here are some properties for which B-definability is unknown:

- Being indecomposable. [B-definable when we restrict to metric continua.]

- Having every interval an indecomposable continuum.

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