Symbolic Dynamics and Dendrites

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Galway 2012, Oxford

Quadratic Maps, Julia Sets and Dendrites

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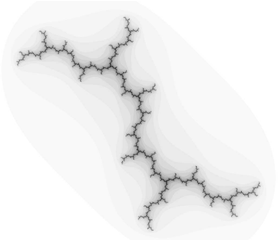
Right is the Julia set for f_i:

$$\left(f_i^2(0) = f_i^4(0) = -1 + i\right).$$



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- If X is **connected**, P_j will be closed for some $0 \leq j \leq n$, in which case It will be discontinuous at points whose orbits visit P_j.

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- Indeed $\mathtt{It}(\mathcal{J}_c)$ is a compact metric space, with a natural arc-length metric d.

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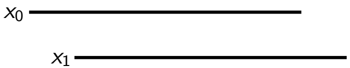

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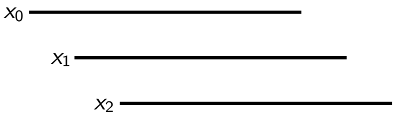

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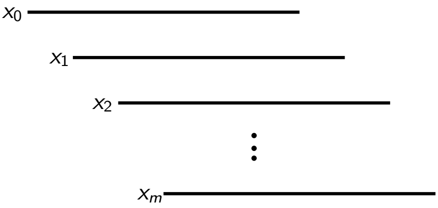

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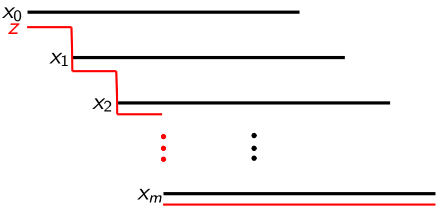

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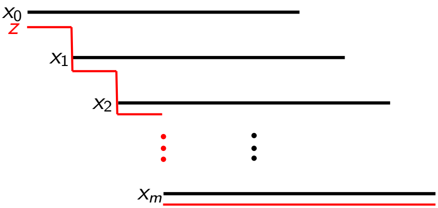

- We can use a similar description for the dendrites $\mathtt{It}(\mathcal{J}_c)$ under the topology $\mathcal{T}^*$; the obstruction is that $\{*\}$ is not open in $\mathcal{T}^*$.

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- Using this method, we can prove the following:

Theorem 3 (Barwell, Raines, 2012)

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- Quadratic maps on the interval (such as the logistic map) give a natural example for when these results do not hold.

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