WEAK CONTRACTIONS IN PARTIAL METRIC SPACES

by

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Partial metric space was introduced by S. G. Matthews in 1994: (*Partial Metric Topology, i n: Proceedings of the 8th Summer Conference on Topology and its Applications , vol . 728, Annals of The Newyork Academy of Sciences , 1994, pp. 183–197.*)

Definition 1. Let X be a nonempty set and let $p : X \times X \to \mathrm{R}^+$ be such that the following are satisfied. For all $x, y, z \in X$

$(PM1)$ $x = y \Longleftrightarrow p(x, x) = p(y, y) = p(x, y)$

$(PM2)$ $p(x, x) \leq p(x, y)$

$(PM3)$ $p(x, y) = p(y, x)$

$(PM4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then the pair (X, p) is called a partial metric space and p is called a partial metric on X.
Here $p(x, x)$ need not be zero always. It is clear that, if $p(x, y) = 0$, then from $(PM1)$ and $(PM2)$ $x = y$.

Definition 2 A metric space is a pair $(X, d : X \times X \to R)$ such that, for all $x, y, z \in X$,

$M0 : 0 \leq d(x, y)$,

$M1 : if \;\; x = y \;\; then \;\; d(x, y) = 0$,

$M2 : if \;\; d(x, y) = 0 \;\; then \;\; x = y$,

$M3 : d(x, y) = d(y, x),\ and$

$M4 : d(x, z) \leq d(x, y) + d(y, z)$.

In pseudo- metric spaces $d(x, x) = 0$, but it is possible that $d(x, y) = 0$.

Example 1 Let S^w be the set of all infinite real sequences and S^* be the set of all finite sequences. Let X=(x_i) and Y=(y_i). Let k be the largest integer for which x_i=y_i for all $i < k$. Let $d(x,y) = 2^{-k}$.

Then this defines a partial metric on $S^w \cup S^*$.

For example, $x = (1,3,5,7)$ and $y = (1,3,5)$ then $d(x,x) = 2^{-5}$. and $d(x,y) = 2^{-4}$.

From the computational viewpoint one needs to know how to compute an infinite sequence. Then $\{x_0, x_1, x_2, ..., x_n\}$ is a partially computed version of $\{x_n\}$ while the latter can be termed as totally computed.

Thus the truth of $x = y$ when $x = \{x_n\}$ and $y = \{y_n\}$ can be asserted only to the extent to which they can be computed.

Example 2. $R^- = (-\infty, 0]$ and $R^+ = [0, \infty)$. Consider the function $p : R^- \times R^- \to R^+$ defined by $p(x,y) = -min\{x,y\}$ for any $x, y \in X$. The pair (R^-, p) is a partial metric space. Here the self-distance for any point $x \in R^-$ is its absolute value that is $p(x,x) =\mid x \mid$.

Example 3. Let $p : R^+ \times R^+ \to R^+$ be defined by $p(x,y) = max\{x,y\}$ for any $x, y \in R^+$. Then (R^+, p) is a partial metric space where the self-distance for any point $x \in R^+$ is its value itself.

Example 4. The interval domain.

Let us consider the set $I = \{[a, b] : a \leq b, a, b \in R\}$ of closed intervals in R and define

$p : I \times I - \to R^{+}$ by setting $p([a, b], [c, d]) = max\{b, d\} - min\{a, c\}$. Then $(I,\, p)$ is a partial metric space.

For any $a, b, c, d, e, f \in R$,

PM1. One verifies that $max\{b, d\} - min\{a, c\} \geq b - a$, hence $p([a, b], [c, d]) \geq p([a, b], [a, b])$,

PM2. suppose that $p([a, b], [a, b]) = p([a, b], [c, d]) = p([c, d], [c, d])$.

Then $b - a = d - c = max\{b, d\} - min\{a, c\}$. So $[a, b]$ and $[c, d]$ have the same length. Suppose that $max\{b, d\} = b$, then $min\{a, c\} = c$ thus $[a, b] \subset [c, d]$. Since they have the same length , they must be equal, that is $[a, b] = [c, d]$.

PM3. It is clear that $p([a, b], [c, d]) = ([a, b], [c, d])$.

PM4. Consider $p([a, b], [e, f]) + p([e, f], [c, d]) - p([e, f], [e, f]) = max\{b, f\} - min\{a, e\} + max\{f, d\} - min\{c, e\} - f + e$. One verifies that $max\{b, f\} + max\{f, d\} - f \geq max\{b, d\}$ and $-min\{a, e\} - min\{e, c\} - e \geq -min\{a, c\}$, hence we have that

$p([a, b], [c, d]) \leq p([a, b], [e, f]) + p([e, f], [c, d]) - p([e, f], [e, f])$.

The self- distance $p([a, b], [a, b])$ for any $a, b \in R, a \leq b$ is the length $b - a$ of the interval $[a, b]$.

Here, $[a, b] \sqsubseteq [c, d]$ if , and only if, $[c, d] \subseteq [a, b]$. Indeed, $p([a, b], [a, b]) = p([a, b], [c, d])$ implies $b - a = max\{b, d\} - min\{a, c\}$. Suppose that $d > b$, then $b - a = d - min\{a, c\}$ and $min\{a, c\} - a = d - b > 0$, hence $min\{a, c\} > a$ which is impossible, then $d \leq b$. Similarly, one proves that $a \leq c$, otherwise $b - a = max\{b, d\} - c$ implies $b > max\{b, d\}$ which is impossible.

Consequently, $[c,d] \subseteq [a,b]$ if and only if $p([a,b],[a,b]) = p([a,b],[c,d])$.

There are good number of works exploring the structure of partial metric spaces, some of these are the following:

1. R. Heckmann, Applied Categorical Structures 7 (1999), 71—83.

2. S. Romagueraand et al, Appl. General Topology, 3 (2002) 91 –112.

3. M. P. Schellekens , Theoretical Computer Science , 315 (2004), no:1 , 135 –149

.4. I. Altun , et al, Topology and Appl. 157 (2010), no: 18, 2778– 2785
.

5. D. Ili´c, et al, Appl. Math. Lett ., 24 (2011),n o:8, 1326 —1330.

6. T. Abdeljawad, et al; Comput. Math. Appl. 63 (2012), no:3 , 716 –719 .

For each partial metric space (X,p) let $\sqsubseteq_p$ be the binary relation over X such that $x \sqsubseteq_p y$ if and only if $p(x,x) = p(x,y)$. Then it can be shown that $(X, \sqsubseteq_p)$ is a partially ordered set.

Referring to example 1, $x \sqsubseteq_p y$ if and only if either $x_i = y_i$ for all i or there exists some $k < \infty$ such that the length of x is k and for each $i \leq k$, $x_i = y_i$. In other words, $x \sqsubseteq_p y$ if and only if x is an initial part of y . For finite sequences we have an example:

$$\langle\rangle \sqsubseteq_p \langle 2\rangle \sqsubseteq_p \langle 2,3\rangle, \sqsubseteq_p \langle 2,3,5\rangle \sqsubseteq_p ...,$$

whose least upper bound is the infinite sequence $\{2, 3, 5, ...\}$of all prime numbers.

The family $\{B^p_\varepsilon(x) : x \in X, \ \ \varepsilon > 0\}$ where $B^p_\varepsilon(x) = \{y : p(x, y) < p(x, x) + \varepsilon\}$

is a basis for a topology τ. This topology is a T_0 topology.

The family $\{B^{p^*}_\varepsilon(x) : x \in X, \ \ \varepsilon > 0\}$ where $B^{p^*}_\varepsilon(x) = \{y : p(x, y) < p(x, x) + \varepsilon\}$ is a basis for the another topology τ^*.

Thus we have a bi-topological space (X, τ, τ^*).

There is a symmetrization topology $\tau^s = \tau \vee \tau^*$.

A partial metric induces a quasi metric given by $q(x, y) = p(x, y) - p(x, x)$. It has its dual $q^*(x, y) = q(y, x) = p(x, y) - p(y, y)$ Then the symmetrization is $d_p(x, y) = q^*(x, y) + q(y, x) = 2p(x, y) - p(x, x) - p(y, y)$, which is a metric .

The topology $\tau \vee \tau^*$ is actually the metric topology induced by the above metric.

The relation can be viewed topologically

$x \sqsubseteq_p y$ iff $p(x, x) = p(x, y)$ ($x \sqsubseteq_p y$ often called "is part of").

Equivalently, $x \sqsubseteq_p y$ iff $y \in B_\varepsilon(x)$ for each $\varepsilon > 0$.

iff $x \in cl(y)$.

The relation $x\rho y$ iff $x \in cl(y)$ is automatically reflexive and transitive.

A topological space is $T_0 \Longleftrightarrow x \in cl(y)$ and $y \in cl(x)$ only when $x = y$.

From here we can also conclude that the topology is T_0.

Definition 3 Let (X, p) be a partial metric space.

A sequence $\{x_n\}$ in the partial metric space (X, p) converges to the limit x if and only if

$$\lim_{n\to\infty} p(x, x_n) = p(x, x).$$

Suppose that (x_n) is a sequence in a partial metric space (X, p), we define $L(x_n)$ to be the set of limit points of (x_n).

As an example , in the usual partial metric (R^-, p), the sequence $(-\frac{1}{n})$ has $L(-\frac{1}{n}) = (-\infty, 0)$.

Suppose that $x < 0$, then $p(-\frac{1}{n}, x) = -min\{-\frac{1}{n}, x\}$. Let $\varepsilon > 0$ be arbitrary. Then there exists $N \geq 1$ such that $-\frac{1}{n} > x$, hence $min\{-\frac{1}{n}, x\} = x$ and $p(-\frac{1}{n}, x) = -x = p(x, x) < p(x, x) + \epsilon$. Consequently $-\frac{1}{n} \in B_\varepsilon(x)$, hence $(-\frac{1}{n}\ )$ converges to x

Proposition 1 Let (x_n) be a sequence in a partial metric space (X, p). If a point $a \in L(x_n)$ and $a^/ \sqsubseteq a$, then $a^/ \in L(x_n)$.

Definition 4 A sequence $\{x_n\}$ in the partial metric space (X, p) is called a Cauchy sequence if $\lim\limits_{m,n\to\infty} p(x_m, x_n)$ exists and is finite.

Definition 5 A partial metric space (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ to a point $x \in X$ such that $p(x, x) = \lim\limits_{m,n\to\infty} p(x_m, x_n)$.

Definition 6 A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0, \delta)) \subseteq B_p(fx_0, \varepsilon)$.

The following implication follows from the above definition.

If a function $f : X \to X$ where (X, p) is a partial metric space is continuous then $fx_n \to fx$ whenever $x_n \to x$ as $n \to \infty$.

Lemma 2 Let (X, p) be a partial metric space.

(1) A sequence $\{x_n\}$ is a Cauchy sequence in the partial metric space (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).
(2) A partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete. Moreover, $\lim_{n\to\infty} d_p(x, x_n) = 0$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{m,n\to\infty} p(x_m, x_n)$.

Theorem 1 Banach contraction mapping theorem

Let (X, d) be a complete metric space and T be a self mapping on X satisfying the following conditions:

$$d(Tx, Ty) \leq q \cdot d(x, y) \text{ for all } x, y \in X \text{ and } q \in (0, 1).$$

Then T has a unique fixed point in X.

In functional analysis we find a lot of efforts to generalize the Banach's contraction mapping principle. Some references are

1. D.W.Boyd et al , Proc. Amer. Math Soc, 20 (1969) 458-464.
2. M.A.Geraghty,Proc.Amer.Math.Soc.40 (1973) 604–608.
3. J. Merryeld et al, Proc. Amer. Math. Soc. 130 (4) (2002) 927-933.
4. A.D. Arvanitakis et al , Proc. Amer. Math. Soc., 131 (12) (2003) 3647-3656.
5. T. Suzuki , Proc. Amer. Math. Soc. 136 (5) (2008) 1861-1869.

Alber and Guerre-Delabriere introduced the concept of weak contraction in Hilbert spaces (Ya. I. Alber, et al , New Results in Operator Theory and its Applications, in: Oper.Theory Adv.Appl.,vol.98, Birkhä user,Basel,1997, pp.7-22.). Rhoades in [Nonlinear Anal. , 47(4) (200l), 2683-2693.] has shown that the result which Alber et al. proved is also valid in complete metric spaces. We state the result of Rhoades in the following:

Definition 7 (weakly contractive mapping) A mapping $T : X \to X$ where (X, d) is a metric space is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)),$$

where $x, y \in X$ and $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$ when $\phi(t) = (1 - k)t$, it reduces to a contradiction.

Theorem 2 If $T : X \to X$ is a weakly contractive mapping where (X, d) is a complete metric space, then T has a unique fixed point.

Dutta and Choudhury [Fixed Point Theory and Application (2008), Article Id06368, 8pages.] proved a generalization employing a method different from that used by Rhoades.

Theorem 3 Let (X, d) be a complete metric space and $T : X \to X$ be a self mapping satisfying the inequality

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)),$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continues and monotone non decreasing function with $\psi(t) = 0 = \phi(t)$ if and only if t=0. Then T has a unique fixed point.

ψ is altering distance function which is a control function. Some works on control functions are

S.V.R. Naidu, Czechoslovak Mathematical Journal, 53(128), (2003),205-212.

K.P.R. Sastry et al, Ind. J. Pure. Appl. Math., 30 (6), (1999), 641-647.

B.S. Choudhury et al, Soochow J. Math., 31(1), (2005), 71-81.

D. M ihet¸, Nonlinear Anal., 71 (2009), 2734- 2738.

Some other works on weak contraction are

1. D. Dori´c, Appl. Math. Lett., 22 (2009), 1896-1900.
2. C. E. Chidume et al, J. Math. Anal. Appl., 270(1) (2002), 189-199.
3. Choudhury et al, NonlinearAnalysis72(2010)1589 1593
4. Choudhury et al, Nonlinear Anal. 74 (2011) 2116–2126
5. Choudhury et al, J. Nonlinear Sci. Appl. 5 (2012), 243-251

There are two further generalisations of weak contractions using two different technnique:

1. O. Popescu, Fixed points for (ψ, ϕ)- weak contractions, Appl. Math. Lett., 24 (2011) 1-4.

2. B. S. Choudhury A. Kundu, (ψ, α, β)-weak contractions in partially ordered metric spaces, Appl. Math. Lett. 25 (2012) 6-10.

Fixed point studies in partial metric spaces were initiated in its introductory paper.

Some subsequent works:

1. Karapiner et al, Applied Math. Lett. 24 (2011) 1894-1899.
2. Altun et al, Topology and its Appl. 157 (2010) 2778-2785.
3. I. D. Itic, Appl. Math. Lett. 24 (2011) 1326-1330.
4. O. Valero, Appl.Gen. Topology, 6 (2005) 229-240.

1 Main Results

Theorem 2.1 Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is a complete partial metric space. Suppose that ψ, α, $\beta : [0, \infty) \to [0, \infty)$ are such that, ψ is continuous and monotone non-decreasing, α is continuous, β is lower semi-continuous, with

$$\psi(t) = 0 \text{ if and only if } t = 0,\ \alpha(0) = \beta(0) = 0 \tag{1.1}$$

$$\text{and } \psi(t) - \alpha(t) + \beta(t) > 0 \ \text{ for all } \ t > 0. \tag{1.2}$$

Let $f : X \to X$ be a non-decreasing and continuous mapping such that

$$\psi(p(fx, fy)) \leq \alpha(p(x, y)) - \beta(p(x, y)),\ \text{for all } x, y \in X \text{ with } \ x \preceq y. \tag{1.3}$$

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. By a condition of the theorem there exists $x_0 \in X$ such that $x_0 \preceq fx_0$. We define $x_1 \in X$ as $x_1 = fx_0$, then $x_0 \preceq fx_0 = x_1$. Since f is non-decreasing, it follows that $fx_0 \preceq fx_1$. In this way we construct the sequence $\{x_n\}$ recursively as

$$fx_n = x_{n+1} \ \text{ for all } n \geq 0 \tag{1.4}$$

for which

$$x_0 \preceq fx_0 = x_1 \preceq fx_1 = x_2 \preceq fx_2 \preceq .... \preceq fx_{n-1} = x_n \preceq fx_n = x_{n+1} \preceq ..... \tag{1.5}$$

If $x_n = x_{n+1}$, then f has a fixed point. Therefore we assume that

$$x_n \neq x_{n+1},\ \text{for all } n \geq 0.$$

Then it follows from the definition of p that

$$p(x_n, x_{n+1}) \neq 0 \text{ for all } n \geq 0. \tag{1.6}$$

Let, if possible, for some n

$$p(x_n, x_{n+1}) < p(x_{n+1}, x_{n+2}). \tag{1.7}$$

Substituting $x = x_n$ and $y = x_{n+1}$ in (2.3), using (2.4), (2.5), (2.7) and the monotone property of ψ, for all $n \geq 0$, we have

$$\begin{aligned}\psi(p(x_n, x_{n+1})) &\leq \psi(p(x_{n+1}, x_{n+2})) = \psi(p(fx_n, fx_{n+1})) \\ &\leq \alpha(p(x_n, x_{n+1})) - \beta(p(x_n, x_{n+1})).\end{aligned} \tag{1.8}$$

Then, by (2.2), it follows that $p(x_n, x_{n+1}) = 0$ which contradicts (2.6). Therefore, for all $n \geq 1$, we have

$$p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n).$$

Thus the sequence $\{p(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers and therefore there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r. \tag{1.9}$$

Taking $n \to \infty$ in (2.8), using the lower semi continuity of β and the continuities of ψ and α, we obtain $\psi(r) \leq \alpha(r) - \beta(r)$, which, by (2.2), implies that $r = 0$. Hence

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{1.10}$$

Then, by (P2) of definition 1.1, we obtain

$$\lim_{n\to\infty} p(x_n, x_n) = 0 \text{ and } \lim_{n\to\infty} p(x_{n+1}, x_{n+1}) = 0. \tag{1.11}$$

Since, from (1.1), $d_p(x,y) \leq 2p(x,y)$ for all $x, y \in X$, we have

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0. \tag{1.12}$$

Next we show that $\{x_n\}$ is a Cauchy sequence in (X, d_p). If not, then there exists some $\varepsilon > 0$ for which we can find two sub-sequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that, for all $k \geq 0$,

$$n(k) > m(k) > k,$$

$$d_p(x_{m(k)}, x_{n(k)}) \geq \varepsilon. \tag{1.13}$$

and

$$d_p(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{1.14}$$

Now, for all $k \geq 0$, we have $\varepsilon \leq d_p(x_{m(k)}, x_{n(k)}) \leq d_p(x_{m(k)}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{n(k)})$
$< \varepsilon + d_p(x_{n(k)-1}, x_{n(k)})$ (by (2.14)).

Taking $k \to \infty$ in the above inequality, and using (2.12), we obtain

$$\lim_{k\to\infty} d_p(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{1.15}$$

Also, for all $k \geq 0$, we have
$d_p(x_{m(k)+1}, x_{n(k)+1}) \leq d_p(x_{m(k)+1}, x_{m(k)}) + d_p(x_{m(k)}, x_{n(k)}) + d_p(x_{n(k)}, x_{n(k)+1})$
and $d_p(x_{m(k)}, x_{n(k)}) \leq d_p(x_{m(k)}, x_{m(k)+1}) + d_p(x_{m(k)+1}, x_{n(k)+1}) + d_p(x_{n(k)+1}, x_{n(k)})$.

Taking limit as $k \to \infty$ in the above two inequalities, using (2.12) and (2.15), we have

$$\lim_{k\to\infty} d_p(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{1.16}$$

Again, by (1.1), for all $k > 0$,

$$d_p(x_{m(k)}, x_{n(k)}) = 2p(x_{m(k)}, x_{n(k)}) - p(x_{m(k)}, x_{m(k)}) - p(x_{n(k)}, x_{n(k)}),$$

and

$$d_p(x_{m(k)+1}, x_{n(k)+1}) = 2p(x_{m(k)+1}, x_{n(k)+1}) - p(x_{m(k)+1}, x_{m(k)+1}) - p(x_{n(k)+1}, x_{n(k)+1}).$$

. Taking $k \to \infty$, in the above two relations, using (2.11), (2.15) and (2.16) we get

$$\lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}) = \frac{\varepsilon}{2} \tag{1.17}$$

and

$$\lim_{k\to\infty} p(x_{m(k)+1}, x_{n(k)+1}) = \frac{\varepsilon}{2}. \tag{1.18}$$

Again since $m(k) < n(k)$ implies $x_{m(k)} \preceq x_{n(k)}$, substituting $x = x_{m(k)}$ and $y = x_{n(k)}$ in (2.3), for all $k \geq 0$, we get,

$$\begin{aligned} \psi(p(x_{m(k)+1}, x_{n(k)+1})) &= \psi(p(fx_{m(k)}, fx_{n(k)})) \\ &\leq \alpha(p(x_{m(k)}, x_{n(k)})) - \beta(p(x_{m(k)}, x_{n(k)})). \end{aligned}$$

Taking $k \to \infty$ in the above inequality, using (2.17), (2.18), the continuities of ψ, α and the facts that β is lower semi continuous, we have,

$$\psi(\frac{\varepsilon}{2}) \leq \alpha(\frac{\varepsilon}{2}) - \beta(\frac{\varepsilon}{2}).$$

By (2.2), this implies that $\varepsilon = 0$ which is a contradiction. Therefore the sequence $\{x_n\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, by lemma 1.3, (X, d_p) is also complete and therefore the sequence $\{x_n\}$ is convergent to some z in X, that is,

$$\lim_{n\to\infty} x_n = z. \tag{1.19}$$

Thus by lemma 1.3

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{1.20}$$

Again by (1.1), for all $m, n > 0$,

$$d_p(x_n, x_m) = 2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m).$$

Taking limit $m, n \to \infty$, using (2.11) and the fact that $\{x_n\}$ is a Cauchy sequence in (X, d_p), we have

$$\lim_{n,m \to \infty} p(x_n, x_m) = 0.$$

Then from (2.20), it follows that

$$p(z, z) = \lim_{n \to \infty} p(x_n, z) = \lim_{n,m \to \infty} p(x_n, x_m) = 0. \tag{1.21}$$

Next we prove that $fz = z$.

By virtue of (2.19), the continuity of f implies that $fx_n \to fz$ as $n \to \infty$. Then, by lemma 1.3, we have

$$p(fz, fz) = \lim_{n \to \infty} p(fx_n, fz) = \lim_{n \to \infty} p(x_{n+1}, fz). \tag{1.22}$$

Now,

$$\begin{aligned} p(z, fz) &\leq p(z, x_{n+1}) + p(x_{n+1}, fz) - p(x_{n+1}, x_{n+1}) \\ &\leq p(z, x_{n+1}) + p(x_{n+1}, fz). \end{aligned}$$

Taking $n \to \infty$ in the above inequality, using (2.19), (2.21) and (2.22) we obtain

$$\begin{aligned} p(z, fz) &\leq \lim_{n \to \infty} p(z, x_{n+1}) + \lim_{n \to \infty} p(x_{n+1}, fz) \\ &= p(fz, fz). \end{aligned}$$

Using the above inequality and the monotone property of ψ, we obtain

$$\psi(p(z, fz)) \leq \psi(p(fz, fz)) \leq \alpha(p(z, z)) - \beta(p(z, z)) \qquad \text{(by (2.3))}.$$

Then, form (2.1) and (2.21), we obtain

$$p(z, fz) = 0.$$

It then follows from (P1) and (P2) of the definition 1.1 that $z = fz$.

This completes the proof of the theorem.

Our next theorem is obtained by replacing the continuity of f in theorem 2.1 by an ordered theoretic condition.

Theorem 2.2 Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is a complete partial metric space. We assume that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then

$$x_n \preceq z \text{ for all } n \geq 0. \tag{1.23}$$

Suppose that ψ, α, $\beta : [0, \infty) \to [0, \infty)$ are such that, ψ is continuous and monotone non-decreasing, α is continuous, β is lower semi-continuous, with

$$\psi(t) = 0 \text{ if and only if } t = 0, \ \alpha(0) = \beta(0) = 0$$

$$\text{and } \psi(t) - \alpha(t) + \beta(t) > 0 \ \text{ for all } \ t > 0.$$

Let $f : X \to X$ be a non-decreasing mapping such that

$$\psi(p(fx, fy)) \leq \alpha(p(x, y)) - \beta(p(x, y)) \text{ for all } x, y \in X \text{ and } \ x \prec y \ (x \neq y), \tag{1.24}$$

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. Following the steps identically as in the proof of the theorem 2.1 we obtain (2.19) and (2.21). Then, by (2.6) and (2.19), we have that $\{x_n\}$

is a non-decreasing sequence that converges to z in X. If $x_n = z$, for some n, then, from (2.5) and (2.23), it follows that $x_n = x_{n+1}$, in which case we have a fixed point. So we assume that $x_n \neq z$ for all $n \geq 0$. Then, from (2.24), we obtain

$$\psi(p(fz, x_{n+1})) = \psi(p(fz, fx_n)) \leq \alpha(p(z, x_n)) - \beta(p(z, x_n)). \qquad (1.25)$$

Also,

$$\begin{aligned} p(fz, x_{n+1}) &\leq p(fz, z) + p(z, x_{n+1}) - p(z, z) \\ \text{and } p(fz, z) &\leq p(fz, x_{n+1}) + p(x_{n+1}, z) - p(x_{n+1}, x_{n+1}). \end{aligned}$$

Taking $n \to \infty$, in the above two inequalities, using (2.11) and (2.21) we get

$$p(fz, z) = \lim_{n\to\infty} p(fz, x_{n+1}).$$

Taking $n \to \infty$ in (2.25), using the continuities of $\psi$and α, the lower semi continuity of β, (2.19), (2.21), and the above limit we have

$$\psi(p(fz, z)) \leq \alpha(p(z, z)) - \beta(p(z, z)). \qquad (1.26)$$

In view of (2.1) and (2.21) it then follows that

$$p(fz, z) = 0.$$

Since $p(fz, z) = 0$, using (P2) and (P1) of definition 1.1, we have $z = fz$.

Remark 2.1: In theorem 2.2 we require the inequality in (2.24), which is the same as in (2.3), only to be satisfied for $x \prec y$, while in the proof we have given for theorem 2.1, it is necessary to assume that the inequality also holds when $x = y$.

Theorem 2.3 Let $(X, \preceq)$ be a partially ordered set and let there be a partial metric p on X such that (X, p) is a complete partial metric space. Let $f : X \to X$ be a continuous and non-decreasing mapping such that

$$\psi(p(fx, fy)) \leq \psi(p(x, y)) - \beta(p(x, y)) \text{ whenever } x, y \in X \text{ and } x \preceq y, \tag{1.27}$$

where

i) $\psi : [0, \infty) \to [0, \infty)$ is a monotone non-decreasing function such that $\psi(t) = 0$ if and only if $t = 0$,

ii) $\beta : [0, \infty) \to [0, \infty)$ is a function satisfying $\beta(0) = 0$, $\liminf_{n\to\infty} \beta(a_n) > 0$ whenever $\lim_{n\to\infty} a_n = a > 0$,

iii) $\beta(t) > \psi(t) - \psi(t^-)$ for all $t > 0$, where $\psi(t^-)$ is the left limit of ψ at t.

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. Starting with $x_0 \in X$, and following the same steps as in theorem 2.1, we obtain a sequence $\{x_n\}$ in X defined as

$$fx_n = x_{n+1} \text{ for all } n \geq 0, \tag{1.28}$$

for which

$$x_0 \preceq fx_0 = x_1 \preceq fx_1 = x_2 \preceq fx_2 \preceq \ldots\ldots.. \preceq fx_{n-1} = x_n \preceq fx_n = x_{n+1} \preceq \ldots.. \tag{1.29}$$

If $x_n = x_{n+1}$, then f has a fixed point. Therefore we assume that

$$x_n \neq x_{n+1}, \text{ for all } n \geq 0.$$

Then it follows from the definition of p that

$$p(x_n, x_{n+1}) \neq 0 \ \text{ for all } n \geq 0. \tag{1.30}$$

Let, if possible, for some n

$$p(x_n, x_{n+1}) < p(x_{n+1}, x_{n+2}). \tag{1.31}$$

Substituting $x = x_n$ and $y = x_{n+1}$ in (2.27), using (2.28), (2.29), (2.31) and the monotone property of ψ, for all $n \geq 0$, we have

$$\begin{aligned} \psi(p(x_n, x_{n+1})) &\leq \psi(p(x_{n+1}, x_{n+2})) \\ &= \psi(p(fx_n, fx_{n+1})) \\ &\leq \psi(p(x_n, x_{n+1})) - \beta(p(x_n, x_{n+1})). \end{aligned} \tag{1.32}$$

A consequence of the properties of β given in condition (ii) of the theorem is that $\beta(a) > 0$ for $a > 0$. Then from (2.30), $\beta(p(x_n, x_{n+1})) > 0$. With this, (2.32) leads to a contradiction. Therefore, for all $n \geq 1$,

$$p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n).$$

Thus the sequence $\{p(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers and consequently there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r. \tag{1.33}$$

Suppose that $r > 0$. If there exists n such that $p(x_n, x_{n+1}) = r$, then, by (2.32) we have $\psi(r) \leq \psi(r) - \beta(r)$. Since $\beta(r) > 0$, this is a contradiction. So $p(x_n, x_{n+1}) > r$, for all $n \geq 0$. Then taking limit infimum as $n \to \infty$ in (2.32), using (2.33) and the fact that $\{p(x_n, x_{n+1})\}$ is monotone decreasing, we have

$$\psi(r^+) \leq \psi(r^+) - \lim_{n\to\infty} \inf \beta(p(x_n, x_{n+1})).$$

By virtue of condition (ii), $\lim_{n\to\infty}\inf \beta(p(x_n, x_{n+1})) > 0$. So the above inequality leads to a contradiction. Hence

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{1.34}$$

It follows by (P1) and (P2) of definition 1.1 that

$$\lim_{n\to\infty} p(x_n, x_n) = 0. \tag{1.35}$$

Since from (1.1), $d_p(x, y) \leq 2p(x, y)$ for all $x, y \in X$, for all $n \geq 0$, from (2.34) it follows that

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0. \tag{1.36}$$

Next we show that $\{x_n\}$ is a Cauchy sequence in (X, d_p). If not, then there exists some $\varepsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that, for all $k \geq 0$,

$$n(k) > m(k) > k,$$

$$d_p(x_{m(k)}, x_{n(k)}) \geq \varepsilon \tag{1.37}$$

and

$$d_p(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{1.38}$$

Now, for all $k \geq 0$, we have $\varepsilon \leq d_p(x_{m(k)}, x_{n(k)}) \leq d_p(x_{m(k)}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{n(k)})$

$< \varepsilon + d_p(x_{n(k)-1}, x_{n(k)})$ (by (2.38)).

Taking $k \to \infty$ in the above inequality, using (2.36), we obtain

$$\lim_{k\to\infty} d_p(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{1.39}$$

Also, for all $k \geq 0$, we have

$d_p(x_{m(k)-1}, x_{n(k)-1}) \leq d_p(x_{m(k)-1}, x_{m(k)}) + d_p(x_{m(k)}, x_{n(k)}) + d_p(x_{n(k)}, x_{n(k)-1})$

and $d_p(x_{m(k)}, x_{n(k)}) \leq d_p(x_{m(k)}, x_{m(k)-1}) + d_p(x_{m(k)-1}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{n(k)})$.

Taking limit as $k \to \infty$ in the above two inequalities, using (2.36) and (2.39) we obtain

$$\lim_{k\to\infty} d_p(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{1.40}$$

Putting $x = x_{m(k)}$ and $y = x_{n(k)}$ in (1.1), we have

$$d_p(x_{m(k)}, x_{n(k)}) = 2p(x_{m(k)}, x_{n(k)}) - p(x_{m(k)}, x_{m(k)}) - p(x_{n(k)}, x_{n(k)}).$$

Then taking $k \to \infty$ and using (2.35), (2.36) and(2.39) we get

$$\lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}) = \frac{\varepsilon}{2}. \tag{1.41}$$

Similarly, using (2.35), (2.36) and(2.40) we have

$$\lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)-1}) = \frac{\varepsilon}{2}. \tag{1.42}$$

Next we show that for sufficiently large k, $p(x_{m(k)}, x_{n(k)}) \leq \frac{\varepsilon}{2}$.

If not, then there exists a subsequence $\{k(i)\}$ of $\mathbb{N}$ such that for all $i > 0$,

$$\frac{\varepsilon}{2} < p(x_{m(k(i))}, x_{n(k(i))})). \tag{1.43}$$

In view of (2.29), substituting $x = x_{m(k(i))-1}$ and $y = x_{n(k(i))-1}$ in (2.27), for all $i > 0$, we have

$$\begin{aligned} \psi(p(x_{m(k(i))}, x_{n(k(i))})) &= \psi(p(fx_{m(k(i))-1}, fx_{n(k(i))-1})) \\ &\leq \psi(p(x_{m(k(i))-1}, x_{n(k(i))-1})) - \beta(p(x_{m(k(i))-1}, x_{n(k(i))-1})). \end{aligned} \tag{1.44}$$

Taking limit as $i \to \infty$ in (2.44), using (2.42), (2.43) and the monotone property of ψ, we obtain

$$\psi(\frac{\varepsilon^+}{2}) \leq \psi(\frac{\varepsilon^+}{2}) - \liminf_{i\to\infty} \beta(p(x_{m(k(i))-1}, x_{n(k(i))-1})).$$

But by a property of β, (2.41) implies that $\liminf_{i\to\infty} \beta(p(x_{m(k(i))-1}, x_{n(k(i))-1})) > 0$. Then the above inequality gives a contradiction. Thus for sufficiently large k, $p(x_{m(k)}, x_{n(k)}) \leq \frac{\varepsilon}{2}$.

Again from (1.1) we have

$$d_p(x_{m(k)}, x_{n(k)}) = 2p(x_{m(k)}, x_{n(k)}) - p(x_{m(k)}, x_{m(k)}) - p(x_{n(k)}, x_{n(k)}).$$

Taking $k \to \infty$ and using (2.35) and (2.37), we have $p(x_{m(k)}, x_{n(k)}) \geq \frac{\varepsilon}{2}$. Then the above observation along with (2.41) implies that, there exists a positive integer k_1 such that for all $k \geq k_1$,

$$p(x_{m(k)}, x_{n(k)}) = \frac{\varepsilon}{2}. \tag{1.45}$$

Substituting $x = x_{m(k)}$ and $y = y_{n(k)}$ in (2.27), using (2.29), we obtain

$$\begin{aligned} \psi(p(x_{m(k)+1}, x_{n(k)+1})) &= \psi(p(fx_{m(k)}, fx_{n(k)})) \\ &\leq \psi(p(x_{m(k)}, x_{n(k)})) - \beta(p(x_{m(k)}, x_{n(k)})) \end{aligned} \tag{1.46}$$

Then by (2.45), for all $k \geq k_1$

$$\psi(p(x_{m(k)+1}, x_{n(k)+1})) \leq \psi(\frac{\varepsilon}{2}) - \beta(\frac{\varepsilon}{2}) < \psi(\frac{\varepsilon}{2}). \tag{1.47}$$

Thus, by (2.47), using the monotone property of ψ, for all $k \geq k_1$, we have

$$p(x_{m(k)+1}, x_{n(k)+1}) < \frac{\varepsilon}{2}. \tag{1.48}$$

Taking the limit as $k \to \infty$ in (2.46), using (2.45) and (2.48), we obtain $\psi(\frac{\varepsilon}{2}^-) \leq \psi(\frac{\varepsilon}{2}) - \beta(\frac{\varepsilon}{2})$, which contradicts condition (iii).

Therefore the sequence $\{x_n\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, by lemma 1.3, (X, d_p) is complete and consequently the sequence $\{x_n\}$ is convergent to z in X, that is,

$$\lim_{n\to\infty} x_n = z. \tag{1.49}$$

Thus, by lemma 1.3,

$$p(z,z)=\lim_{n\to\infty}p(x_n,z)=\lim_{n,m\to\infty}p(x_n,x_m). \qquad (1.50)$$

Again by (1.1), for all $m,n\geq 0$

$$d_p(x_n,x_m)=2p(x_n,x_m)-p(x_n,x_n)-p(x_m,x_m).$$

Taking limit $m,n\to\infty$, using (2.35) and the fact that $\{x_n\}$ is a Cauchy sequence in (X,d_p), we have

$$\lim_{n,m\to\infty}p(x_n,x_m)=0.$$

Then, from (2.50), it follows that

$$p(z,z)=\lim_{n\to\infty}p(x_n,z)=\lim_{n,m\to\infty}p(x_n,x_m)=0. \qquad (1.51)$$

Next we prove that $fz=z$.

By virtue of (2.49), the continuity of f implies that $fx_n\to fz$ as $n\to\infty$. Then, by lemma 1.3, we have

$$p(fz,fz)=\lim_{n\to\infty}p(fx_n,fz)=\lim_{n\to\infty}p(x_{n+1},fz). \qquad (1.52)$$

Now,

$$\begin{aligned} p(z,fz) &\leq p(z,x_{n+1})+p(x_{n+1},fz)-p(x_{n+1},x_{n+1}) \\ &\leq p(z,x_{n+1})+p(x_{n+1},fz). \end{aligned}$$

Taking $n\to\infty$ in the above inequality, using (2.49), (2.51) and (2.52), we obtain

$$\begin{aligned} p(z,fz) &\leq \lim_{n\to\infty}p(z,x_{n+1})+\lim_{n\to\infty}p(x_{n+1},fz) \\ &= p(fz,fz). \end{aligned}$$

Using the above inequality and the monotone property of ψ, we obtain,

$$\psi(p(z, fz)) \leq \psi(p(fz, fz)) \leq \psi(p(z, z)) - \beta(p(z, z)) \text{ (by(2.27))}. \quad (1.53)$$

In view of (i), (ii) and (2.51) we obtain $p(z, fz) = 0$. Then from (P1) and (P2) of the definition 1.1, it follows that $z = fz$.

Our next theorem is obtained by replacing the continuity of f by an ordered theoretic condition.

Theorem 2.4 Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is a complete partial metric space. We assume that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then

$$x_n \preceq z \text{ for all } n \geq 0. \quad (1.54)$$

Let $f : X \to X$ be a non-decreasing mapping such that

$$\psi(p(fx, fy)) \leq \psi(p(x, y)) - \beta(p(x, y)) \text{ for all } x, y \in X \text{ and } x \prec y \ (x \neq y), \quad (1.55)$$

where ψ and β satisfies all the condition of theorem 2.3. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. Following the steps identically as in the proof of the theorem 2.3 we obtain (2.49) and (2.51). Then, by (2.29) and (2.49), we have that $\{x_n\}$ is a non-decreasing sequence that converges to z in X. If $x_n = z$, for some n, then, from (2.29) and (2.54), it follows that $x_n = x_{n+1}$, in which case we have a fixed point. So we assume that $x_n \neq z$ for all $n \geq 0$.

Now,

$$\begin{aligned} p(fz, x_{n+1}) &\leq p(fz, z) + p(z, x_{n+1}) - p(z, z) \\ \text{and } p(fz, z) &\leq p(fz, x_{n+1}) + p(x_{n+1}, z) - p(x_{n+1}, x_{n+1}). \end{aligned}$$

Taking $n \to \infty$, in the above two inequalities, using (2.35) and (2.51) we get

$$p(fz, z) = \lim_{n \to \infty} p(fz, x_{n+1}). \tag{1.56}$$

From (2.27), we obtain

$$\psi(p(fz, x_{n+1})) = \psi(p(fz, fx_n)) \leq \psi(p(z, x_n)) - \beta(p(z, x_n)).$$

Taking $n \to \infty$, in the above inequality, using (2.56), the continuity of ψ and lower semi continuity of β, we have

$$\psi(p(fz, z)) \leq \psi(p(z, z)) - \beta(p(z, z)).$$

In view of the properties of (i) and (ii) we arrive at a contradiction, unless $p(fz, z) = 0$. Since $p(z, z) = 0$ and $p(z, fz) = 0$, from (P1) and (P2) of definition 1.1, it follows that $z = fz$.
In the sequel, we present several corollaries which extends several existing results.

Remark 2.2: Under the assumption when partial metric is a metric our theorems 2.1 and 2.2 extends several existing results.

1. If we take $\psi(t) = \alpha(t)$ for all $t > 0$ and $\beta(t)$ is a continuous and nondecreasing mapping, in Theorem 2.1 and 2.2.

a) we obtain an extension of theorem 2.1 and 2.2 of [20] to partially ordered metric spaces.

b) Also we obtain an extension of theorem of Dutta and Choudhury [15] to metric spaces.

2. Theorem 2.1 is an extension of the result of Eslamian and Abkar [17] to a partially ordered metric spaces.

2 Examples

In this section we discuss two illustrative examples.

Example 3.1 We describe the following complete partial metric space. Let X =[0,1] and $p : X \times X \to \mathrm{R}^+$ be defined as $p(x, y) = \max\{x, y\}$. Then $(X, \preceq)$ is a partially ordered set with $x \preceq y$ whenever $x \geq y$.

Let $f : X \to X$ be defined as $fx = x - \frac{1}{2}x^2$ for all $x \in X$.

Then f is a continuous function on X.

Let $x_0 = c > 0$. Then $x_0 \preceq fx_0$.

Let $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ be defined respectively as follows

$$\psi(t) = t, \quad \alpha(t) = t - \frac{1}{4}t^2, \qquad \beta(t) = \frac{t^2}{8}, \text{ for all } t \geq 0.$$

Then ψ, α, β are continuous and $\psi(t) - \alpha(t) + \beta(t) = t - t + \frac{1}{4}t^2 - \frac{t^2}{8} = \frac{t^2}{8} > 0$ for all $t > 0$.

Let $x, y \in X$. Without loss of generality we assume $x \geq y$.

$$\begin{aligned} \text{Then,} \qquad p(fx, fy) &= \max\{x - \frac{1}{2}x^2, y - \frac{1}{2}y^2\} = x - \frac{1}{2}x^2, \\ p(x, y) &= \max\{x, y\} = x \end{aligned}$$

$$
\begin{aligned}
\text{and} \quad \psi(p(fx, fy)) &= x - \frac{1}{2}x^2 \leq x - \frac{3}{8}x^2 \\
&= x - \frac{1}{4}x^2 - \frac{1}{8}x^2 \\
&= \alpha(p(x, y)) - \beta(p(x, y)).
\end{aligned}
$$

Thus all the conditions of Theorem 2.1 are satisfied. Then by an application of theorem 2.1 we have a fixed point of f. Here "0" is a fixed point of f.

Example 3.2 We describe the following complete partial metric space. Let $X = \{0, 1, 2, 3, 4, ...\}$. We define $p : X \times X \to \mathrm{R}^+$ as

$$
p(x, y) = \begin{cases} x + y + 2, \text{if } x \neq y, \\ 1, \text{if } x = y. \end{cases}
$$

Then p is a partial metric on X.

The properties $(P1)$, $(P2)$ and $(P3)$ are directly verified by inspection. We prove $(P4)$ in the following. Let $a, b, c \in X$. If $a \neq c$ then

i) $p(a, c) = a + c + 2 < a + b + 2 + b + c + 2 - 1 = p(a, b) + p(b, c) - p(b, b)$ (if $b \neq a$ and $b \neq c$).

ii) $p(a, c) = a + c + 2 < 1 + a + c + 2 = p(a, b) + p(b, c) - p(b, b)$ (if $b = a$ and $b \neq c$).

If $a = c$ then $p(a, c) = 1 \leq p(a, b) + p(b, c) - 1 = p(a, b) + p(b, c) - p(b, b)$. Thus (P4) is satisfied.

In view of (1.1) the function $d_p : X \times X \to R^+$ defined as

$$
d_p(x, y) = \begin{cases} 2x + 2y + 2, \text{if } x \neq y, \\ 0, \text{if } x = y. \end{cases}
$$

It is a metric on X. We define a partial ordering '$\preceq$ ' in X as $x \preceq y$ if and only if $x \geq y$ and $(x - y)$ is divisible by 2, for all $x, y \in \{2, 3, 4, ...\}$ and

$1 \preceq 0, 2 \preceq 1$.

Let $f : X \to X$ be defined as $\quad fx = \begin{cases} x-2, & \text{if } x \geq 2, \\ 0, & \text{if } \ x = 0, 1. \end{cases}$

Let $\psi, \alpha\ , \beta : [0, \infty) \to [0, \infty)$ be defined as $\psi(t) = t$, for all $t \geq 0$,

$$\alpha(t) = \begin{cases} t + \frac{1}{t}, & \text{for } t > 1, \\ 2t^2, & \text{for } t \in [0, 1] \end{cases} \quad \text{and} \quad \beta(t) = \begin{cases} 1 + \frac{1}{t}, & \text{for } t > 1, \\ 2t, & \text{for } t \in [0, 1]. \end{cases}$$

Also for $x_0 = 10$ we have $fx_0 = 8$ that is, $x_0 \preceq fx_0$, Let $x \neq y$. With out loss of generality we assume that $x > y$. Then the following cases are possible.

Case I $\ x \in \{1, 2\}$ and $y \in \{0, 1, 2\}$, then $fx = 0 = fy$, and

. $\quad p(f(x), f(y)) = p(0, 0) = 1$, $p(1, 0) = p(0, 1) = 3$, $\ p(2, 1) = p(1, 2) = 5$.

Thus $\psi(p(fx, fy)) = 1 \leq \alpha(p(x, y)) - \beta(p(x, y))$ is satisfied.

Case II $\ x \in \{3, 4, 5, ...\}$ and $\ y \in \{0, 1, 2\}$, then $fx = x - 2$, $fy = 0$.

Now $p(f(x), f(y)) = x-2+2 = x$, and $\psi(p(fx, fy)) = x$. Also $p(x, y) = x + y + 2 \geq 5$.

Therefore,

$$\psi(p(fx, fy)) = x < x+y+1 = (x+y+2)+\frac{1}{(x+y+2)}-1-\frac{1}{(x+y+2)}$$
$$= \alpha(p(x, y)) - \beta(p(x, y)).$$

Cases III $x \in \{4, 5, ...\}$ and $\ y \in \{3, 4, 5...\}$, then $fx = x-2$, $fy = y-2$.

Then $\quad p(fx, fy) = x + y - 2$, for $x \neq y\ $ and $p(x, y) = x + y + 2$, for $x \neq y$.

$$\psi(p(fx, fy)) = x + y - 2 < x + y + 1$$
$$= x + y + 2 + \frac{1}{x+y+2} - 1 - \frac{1}{x+y+2} = \alpha(p(x, y)) - \beta(p(x, y)).$$

Combining all the above three cases we conclude that for all $x, y \in X$, $\psi(p(fx, fy)) \leq \alpha(p(x, y)) - \beta(p(x, y))$ holds.

It is observed that ψ, α, β and f satisfy all their required conditions in Theorem 2.2. It follows, by an application of Theorem 2.2, that f has a fixed point. Here "0" is the a fixed point of f.

Remark 3.1: It is seen that the inequality (2.3) is not satisfied when $x = y$. Hence Theorem 2.1 is not applicable to example 3.2.

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