Function Spaces and Local Properties

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15th Topology Galway Colloquium

July 09, 2012

Function Spaces

Let X be a topological space.

$$C(X) = \{f : f : X \to \mathbb{R} \text{ which is continuous}\}$$

Basic Open Set $\forall f \in C(X)$, $\epsilon > 0$ and finite set $F \subset X$, define

$$B(f, F, \epsilon) = \{g : |f(x) - g(x)| < \epsilon \text{ where } x \in F\}$$

Function Space $C_p(X)$ is
$C(X)$ with the point-wise convergence topology.

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Generalized Metric Properties

Definition

M_1-Space is a space with a σ closure-preserving base.

Definition

M_3-Space is a space with a σ cushioned pair-base.

Fact

M_1 implies M_3.

Question

Does M_3 imply M_1?

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More Definitions

- Let $\mathcal{B}$ be a family of subsets of X.

Definition

$\mathcal{B}$ is closure-preserving if $\forall \mathcal{B}' \subseteq \mathcal{B}$

$$\overline{\bigcup \mathcal{B}'} = \bigcup \{\overline{B} : B \in \mathcal{B}'\}$$

- Let $\mathcal{P}$ be a family of pairs of subsets of X, i.e. $(P_1, P_2) \in \mathcal{P}$.

Definition

Let $\mathcal{P}$ is cushioned if $\forall \mathcal{P}' \subseteq \mathcal{P}$,

$$\overline{\bigcup \{P_1 : (P_1, P_2) \in \mathcal{P}'} \subseteq \bigcup \{\overline{P_2} : (P_1, P_2) \in \mathcal{P}'\}$$

- $\mathcal{B}$ ($\mathcal{P}$) is σ-closure preserving (cushioned) if it is a countable union of closure preserving (cushioned) collections.

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Local Properties

Definition

X is called a $(\sigma-)m_1$ space
if X has a $(\sigma-)$closure preserving base at every $x \in X$.

Definition

X is called a $(\sigma-)m_3$ space
if X has a $(\sigma-)$cushioned pair-base at every $x \in X$.

Lemma

First countable $\implies$ m_1*-property* $\implies$ m_3*-property, and*
m_i*-properties* $\implies$ $\sigma - m_i$*-properties.*
Monotonically normal $\implies$ m_3*-property.*

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Questions

Theorem

$C_p(X)$ is first countable $\iff$ X is countable.

Theorem (Gartside)

$C_p(X)$ is monotonically normal $\iff$ X is countable.

Question (Dow, Ramírez Martínez, Tkachuk)

$C_p(X)$ is m_3 (or m_1) $\iff$ X is countable?
What if X is compact?

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What if X is compact?

Restrictions on Spaces with $C_p(X)$ having σ-m_3-property

Definition

A space Z is functionally countable if every continuous $f : Z \to \mathbb{R}$ has countable image.

Theorem

If $C_p(X)$ is a σ-m_3 space, then X is functionally countable.

Corollary

If X is compact and $C_p(X)$ is a σ-m_3 space, then X is scattered.

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Compact Scattered Spaces

Let X be a compact scattered space with height $\alpha + 1$, i.e. $X^{(\alpha+1)} = \emptyset$.
Here $X^{(0)}$ be the set of isolated points of X, inductively,

- $X^{(\beta)} = X \setminus X^{(\beta-1)}$ if β is successive;
- $X^{(\beta)} = X \setminus (\bigcup\{X^{(\gamma)} : \gamma < \beta\}$ if β is a limit ordinal;

Theorem

Let X be a compact scattered space with finite height $a + 1$. For each $b < a$, $\exists \mathcal{U}_b = \{U_x : x \in X^{(\leq b)}$ and $\{x\} = U_x \cap X^{(b)}\}$ which is locally finite.

Then $C_p(X)$ is $\sigma - m_1$.

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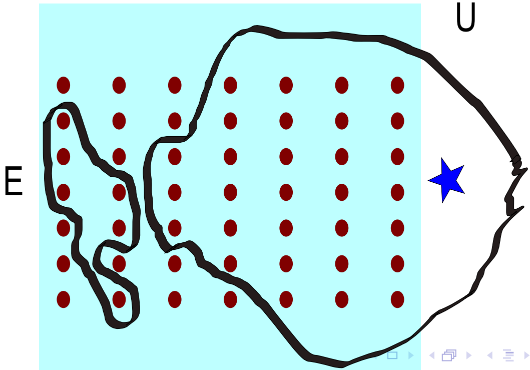
Compact Scattered Spaces

- Let X be a compact scattered space with $X^{(\alpha)} = \{*\}$
- $C_p(X)$ is σ-m_1, if

$$\exists \mathcal{F} = \bigcup \mathcal{F}_n \text{ cofinal in } ([X]^{<\omega}, \subseteq) \text{ such that}$$

$\forall n \forall$ closed neighborhood C of $*$, $\exists E \in [X]^{<\omega}$ and $E \cap C = \emptyset$ such that

$$\forall F \in \mathcal{F}_n \text{ either } F \subseteq C \text{ or } F \cap E \neq \emptyset$$



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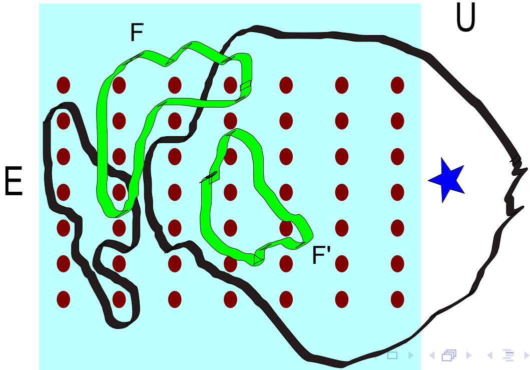

Examples

- $C_p(A(\omega_1))$ and $C_p(A(\omega_1) \oplus \omega)$ are σ-m_1.
- Let $L(\omega_1)$ be one point Lindelofication of discrete space ω_1 $C_p(L(\omega_1))$ is not σ-m_1.

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Compact and Scattered Spaces

Similarly, We can prove,

- Let X be a compact scattered space with $X^{(\alpha)} = \{*\}$
- $C_p(X)$ is $\sigma - m_3$, if and only if

$\exists \mathcal{F} = \bigcup \mathcal{F}_n$ cofinal in $([X]^{<\omega}, \subseteq)$ and $G : \mathcal{F} \to [X]^{<\omega}$ such that $\forall n \forall$ closed neighborhood C, $\exists E \in [X]^{<\omega}$ and $E \cap C = \emptyset$ such that

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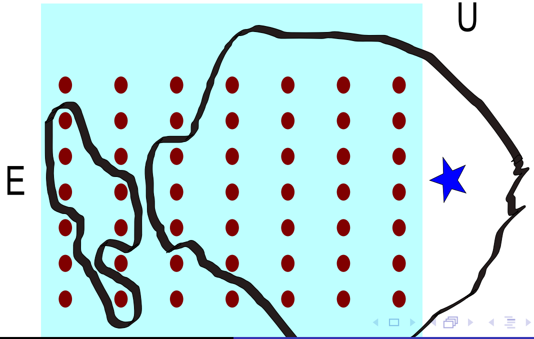

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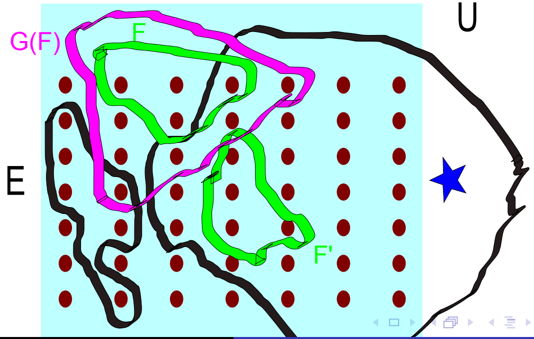

Restrictions on Spaces with $C_p(X)$ having m_3-property

Theorem

If $C_p(X)$ is a m_3 space, then X does not contain an uncountable set of isolated points.

Corollary

If X is compact and $C_p(X)$ is a m_3 space, then X is scattered and separable.

Example

$C_p(A(\omega_1))$ is not a m_3-space.

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Spaces with $C_p(X)$ has m_1 property

Let X be a compact separable scattered space with $X^{(\alpha)} = \{\star\}$.

Theorem (†)

$C_p(X)$ has m_1 property if
$\exists \mathcal{F}$ cofinal in $([X]^{(<\omega)}, \subseteq)$ which satisfies
$\forall$ closed neighborhood C of $\star$, $\exists$ finite set $E \subseteq X \setminus C$
such that $\forall F \in \mathcal{F}$, $E \cap F \neq \emptyset$ or $F \subseteq C$.

Proof of Theorem †

- Let D be the countable dense subset of X.
- Fix $\phi : \mathbb{N} \to D$.
- Define $\mathcal{B} = \{B(0, \{\phi(i), 1 \leq i \leq |F|\} \cup F, 1/(2|F|)) : F \in \mathcal{F}\}$.
- We can verify $\mathcal{B}$ is a closure-preserving base of $C_p(X)$.

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Example From Ψ Space

- Let $\mathcal{A}$ be a Maximal Almost Disjoint family on $\mathbb{N}$;
- Let $\Psi = \mathbb{N} \cup \mathcal{A}$.
  Points in $\mathbb{N}$ are isolated;
  A basic neighborhood of $A \in \mathcal{A}$ is $\{A\} \cup (A \setminus F)$ where $F \subseteq \mathbb{N}$ is finite;
- Let K be the one point compactifcation of Ψ. Let $\star$ be the 'point at infinity';
- $C_p(K)$ has m_1 property.

This answers the question raised by Dow, Ramírez Martínez and Tkachuk.

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Thank You!