Almost Injectivity

Shari S. Levine

15^{th} Galway Topology Colloquium, University of Oxford

11 July, 2012

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Can you identify the person in the following picture?

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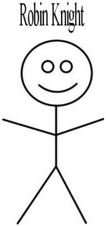

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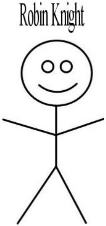

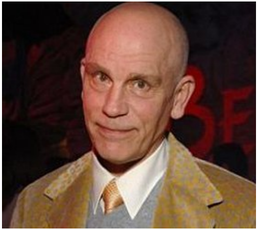

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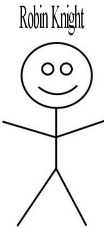

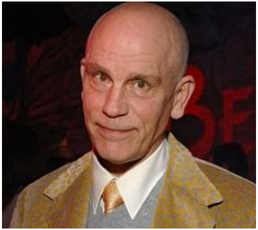

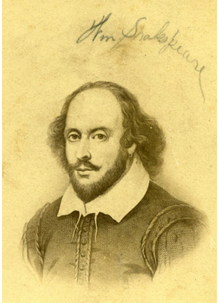

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Can we weaken the injectivity of f so that the theorem remains true?

Other attempts

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1. My entire thesis

Other attempts

1. My entire thesis
2. Cleavability

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2. Cleavability

Definition

Let X and Y be topological spaces. We say X is **cleavable over** Y if for every $A \subseteq X$ there exists a continuous function $f_A : X \to Y$ such that $f_A(A) \cap f_A(X \setminus A) = \emptyset$.

Cleavability

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- “On Cleavability of Topological Spaces over $\mathscr{R}$, $\mathscr{R}^n$, and $\mathscr{R}^\omega$”: every compact Hausdorff space X cleavable over $\mathscr{R}$ (resp. $\mathscr{R}^\omega$) is homeomorphic to a subspace of $\mathscr{R}$ (resp. $\mathscr{R}^\omega$).

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- "On Cleavability of Continua over LOTS": any continuum cleavable over a LOTS L is embeddable into L.
- "Cleavability of compacta over the two arrows": any compactum cleavable over the Double Arrow Space is homeomorphic to a subspace of the Double Arrow Space.

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- "On Cleavability of Continua over LOTS": any continuum cleavable over a LOTS L is embeddable into L.
- "Cleavability of compacta over the two arrows": any compactum cleavable over the Double Arrow Space is homeomorphic to a subspace of the Double Arrow Space.
- "Cleavability and scattered sets of non-trivial fibers": if X is a compactum cleavable over a separable LOTS Y such that there exists a function $f : X \to Y$ with a scattered set of non-trivial fibers, then X us a LOTS

Almost-injectivity

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If X is an infinite T_2 compactum, Y a LOTS, and there exists $f \in \mathscr{C}(X, Y)$ such that $|M_f| \leq \aleph_0$, then is X a LOTS?

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Question

If X is an infinite T_2 compactum, Y a LOTS with property P, and there exists $f \in \mathscr{C}(X, Y)$ such that $|M_f| \leq \aleph_0$, must X have property P?

Two notes

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Example

Let $X = \{x_1, x_2\}$ with the trivial topology, and let $Y = \{y_1, y_2\}$ with the discrete topology. Then there exists an almost-injective function from X to Y (in particular, the constant function onto y_1) but even though Y is T_2, X is not.

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2. We consider any constant function from a countable space X to a LOTS Y to be an almost-injective function.

Two notes proof

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Theorem

If X is compact and T_2, Y is a topological space, and $f : X \to Y$ is a constant map, then X is a LOTS.

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If X is compact and T_2, Y is a topological space, and $f : X \to Y$ is a constant map, then X is a LOTS.

Proof.

X must be countable as $X = M_f$. Since X is compact, countable, and metrizable, by a theorem of Mazurkiewicz & Sierpinski, X must be homeomorphic to a countable ordinal, and is therefore a LOTS. □

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Assume for a contradiction that X contains a dense-in-itself subset A. Let A be closed. Consider $f(A)$.

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Assume for a contradiction that X contains a dense-in-itself subset A. Let A be closed. Consider $f(A)$. Since Y is scattered, $f(A)$ contains an isolated point, y. We know since M_f is countable, that $f^{-1}(y)$ must be countable, and as f is continuous, must be closed and compact as well. Therefore by a theorem of Mazurkiewicz & Sierpinski, $f^{-1}(y)$ is homeomorphic to a countable ordinal. (Note that $f^{-1}(y)$ may be a single point.)

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Other results

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If X is compact T_2, Y is a first-countable T_2 space, and there exists an almost-injective $f \in \mathscr{C}(X, Y)$, then X is first-countable.

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If X is compact T_2, Y is a metric space, and there exists an almost-injective $f \in \mathscr{C}(X, Y)$, then X is metrizable.

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Theorem

If X is compact T_2, Y is T_2, and there exists an almost-injective $f \in \mathscr{C}(X, Y)$, then $|X| \leq |Y|$.

First Counter-Example

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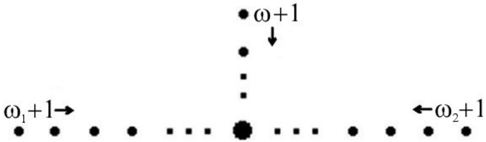

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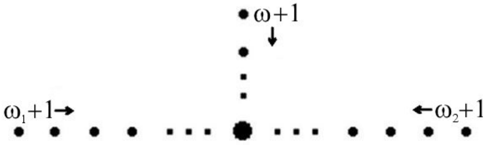

Example

Let X be the one-point compactification of the disjoint union of ω_1, ω_2, and ω, and Y be the one-point compactification of the disjoint union of ω_1 and ω_2. Then there exists an almost-injection from X to Y, but X is not a LOTS.

Second Counter-Example

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Example

Let X be the one-point compactification of the disjoint union of ω_1, ω_2, and ω, and let Y be the one-point compactification of ω_1 and ω_2. Let $\hat{X}$ be the one-point compactification of ω-many copies of X, and let $\hat{Y}$ be the one-point compactification of ω-many copies of Y. Then there exists an almost-injection $f \in \mathscr{C}(\hat{X}, \hat{Y})$, but X is not a LOTS.

Picking up the pieces

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Question

If X is an infinite T_2 compactum, Y a LOTS, and there exists $f \in \mathscr{C}(X, Y)$ such that $|M_f| \leq \aleph_0$, then must X be the union of a LOTS and a scattered set?

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Despite these counter-examples, we feel answering whether the existence of an almost-injective function implies linear orderability of a space is still a worthwhile question to ask.

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Despite these counter-examples, we feel answering whether the existence of an almost-injective function implies linear orderability of a space is still a worthwhile question to ask.
Thus, we begin by looking at almost-injective functions from infinite T_2 compacta to the ordinals.

The basics

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If X is an infinite T_2 compactum, Y is an ordinal, and there exists an almost-injective $f \in \mathscr{C}(X, Y)$, we know:

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If X is an infinite T_2 compactum, Y is an ordinal, and there exists an almost-injective $f \in \mathscr{C}(X, Y)$, we know:

1. X is scattered
2. ...That's about it

The balance between overview and interesting details

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Definition

For ordinal numbers α, the α-th **Cantor-Bendixson derivative** of a topological space X is defined by transfinite induction as follows:

- $X^0 = X$
- $X^{\alpha+1} = (X^\alpha)'$
- $X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha$ for limit ordinals λ.

The smallest ordinal α such that $X^{\alpha+1} = X^\alpha$ is called the **Cantor-Bendixson rank** of X, written as $\mathrm{CB}(X)$.

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Let X be a scattered topological space, and $x \in X$. We use the notation $\mathrm{rank}(x)$ to mean the least ordinal α such that $x \notin X^\alpha$.

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The idea is to consider $\mathrm{CB}(f(M_f))$.

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The idea is to consider $\mathrm{CB}(f(M_f))$. Since M_f is countable, $f(M_f)$ must be countable, and therefore $\mathrm{CB}(f(M_f))$ must be countable. The case where $\mathrm{CB}(f(M_f)) = \alpha + 1$ for $\alpha \neq 0$ is easy. Enumerate the elements of $f(M_f)^\alpha$ as y_n; since Y is an ordinal, and $\mathrm{rank}(y_n) = \alpha + 1$, it must be the case that $cf((0, y_n))$ is countable.

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The case where $\mathrm{CB}(f(M_f)) = \alpha + 1$ for $\alpha \neq 0$ is easy. Enumerate the elements of $f(M_f)^\alpha$ as y_n; since Y is an ordinal, and $\mathrm{rank}(y_n) = \alpha + 1$, it must be the case that $cf((0, y_n))$ is countable. Therefore we can split up Y into easy to manage intervals, use the inductive hypothesis, and presto - we get a LOTS.
The case where $\mathrm{CB}(f(M_f)) = \lambda$ for some limit ordinal λ is a bit trickier, but has the same principle. We just consider $(\overline{f(M_f)})^\lambda$ instead of $f(M_f)^\lambda$. □

cont'd.

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In any case, however, we still get a LOTS! □

Open Questions

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If X is an infinite T_2 compactum, Y is a LOTS, and there exists an almost-injective $f \in \mathscr{C}(X, Y)$, must X be the union of a LOTS and a scattered set?

References

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