The ultra-quasi-metric injective hull of a $T_0$-ultra-quasi-metric space

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Recently Kemajou, Kunzi and Otafudu have developed a concept of hyperconvexity (called Isbell-convexity) that is appropriate in the category of $T_0$-quasi-metric spaces and non-expansive maps. In particular an explicit construction of the corresponding hull (called Isbell-hull) was provided.
In this talk we shall discuss how the investigations of Kemajou, Kunzi and Otafudu can be modified in order to obtain a theory that is suitable for $T_0$-ultra-quasi-metric spaces.
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Let \( X \) be a set and \( u : X \times X \to [0, \infty) \) be a function mapping into the set \([0, \infty)\) of non-negative reals. Then \( u \) is an \textit{ultra-quasi-pseudometric} on \( X \) if
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(i) $u(x, x) = 0$ for all $x \in X$, and

(ii) $u(x, z) \leq \max\{u(x, y), u(y, z)\}$ whenever $x, y, z \in X$. 

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Note that the so-called conjugate $u^{-1}$ of $u$, where $u^{-1}(x, y) = u(y, x)$ whenever $x, y \in X$, is an ultra-quasi-pseudometric, too.
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Note that the so-called \textit{conjugate} $u^{-1}$ of $u$, where $u^{-1}(x, y) = u(y, x)$ whenever $x, y \in X$, is an ultra-quasi-pseudometric, too. The set of open balls $\{\{y \in X : u(x, y) < \epsilon\} : x \in X, \epsilon > 0\}$ yields a base for the topology $\tau(u)$ \textit{induced by $u$ on $X$}. 

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If \( u \) also satisfies the condition

(iii) for any \( x, y \in X, \) \( u(x, y) = 0 = u(y, x) \) implies that \( x = y \), then \( u \) is called a \textit{T}_0\textit{-ultra-quasi-metric}. 
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Let \( X \) be a set and \( u : X \times X \to [0, \infty) \) be a function mapping into the set \( [0, \infty) \) of non-negative reals. Then \( u \) is an **ultra-quasi-pseudometric** on \( X \) if

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Observe that then \( u^s = u \vee u^{-1} \) is an **ultra-metric** on \( X \).
Example

Let $X = [0, \infty)$ be equipped with $n(x, y) = x$ if $x, y \in X$ and $x > y$, and $n(x, y) = 0$ if $x, y \in X$ and $x \leq y$. It is easy to check that $(X, n)$ is a $T_0$-ultra-quasi-metric space.
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Note also that for $x, y \in [0, \infty)$ we have $n^s(x, y) = \max\{x, y\}$ if $x \neq y$ and $n(x, y) = 0$ if $x = y$. Observe that the ultra-metric $n^s$ is complete on $[0, \infty)$. 
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Lemma

Let $a, b, c \in [0, \infty)$. Then the following conditions are equivalent:

(a) $n(a, b) \leq c$.
(b) $a \leq \max\{b, c\}$.
Corollary

Let \((X, u)\) be an ultra-quasi-pseudometric space. Consider a map \(f : X \to [0, \infty)\) and let \(x, y \in X\). Then the following are equivalent:

(a) \(n(f(x), f(y)) \leq u(x, y)\);
(b) \(f(x) \leq \max\{f(y), u(x, y)\}\).
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(a) \(n(f(x), f(y)) \leq u(x, y)\);
(b) \(f(x) \leq \max\{f(y), u(x, y)\}\).

Corollary

Let \((X, u)\) be an ultra-quasi-pseudometric space.

(a) Then \(f : (X, u) \to ([0, \infty), n)\) is a contracting map if and only if \(f(x) \leq \max\{f(y), u(x, y)\}\) whenever \(x, y \in X\).
(b) Then \(f : (X, u) \to ([0, \infty), n^{-1})\) is a contracting map if and only if \(f(x) \leq \max\{f(y), u(y, x)\}\) whenever \(x, y \in X\).
Definition

Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space and let \(\mathcal{FP}(X, u)\) be the set of all pairs \(f = (f_1, f_2)\) of functions where \(f_i : X \to [0, \infty)\) \((i = 1, 2)\).
Definition

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\[
N(((f_1, f_2), (g_1, g_2)) = \max\{\sup_{x \in X} n(f_1(x), g_1(x)), \sup_{x \in X} n(g_2(x), f_2(x))\}.
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It is obvious that \( N \) is an extended \( T_0 \)-ultra-quasi-metric on the set \( \mathcal{FP}(X, u) \) of these function pairs.
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Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. We shall say that a pair \(f \in \mathcal{FP}(X, u)\) is **strongly tight** if for all \(x, y \in X\), we have

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u(x, y) \leq \max\{f_2(x), f_1(y)\}.
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The set of all strongly tight function pairs of a \(T_0\)-ultra-quasi-metric space \((X, u)\) will be denoted by \(UT(X, u)\).
Lemma

Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. For each \(a \in X\),
\[ f_a(x) := (u(a, x), u(x, a)) \text{ whenever } x \in X, \]
is a strongly tight pair belonging to \(UT(X, u)\).
Lemma

Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. For each \(a \in X\),
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Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. We say that a function pair \(f = (f_1, f_2)\) is minimal among the strongly tight pairs on \((X, u)\) if it is a strongly tight pair and if \(g = (g_1, g_2)\) is strongly tight on \((X, u)\) and for each \(x \in X\), \(g_1(x) \leq f_1(x)\) and \(g_2(x) \leq f_2(x)\), then \(f = g\).
Lemma

Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. For each \(a \in X\), 
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Minimal strongly tight function pairs are also called \textit{extremal strongly tight}
function pairs.
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By \(\nu_q(X, u)\) (or more briefly, \(\nu_q(X)\)) we shall denote the set of all minimal
strongly tight function pairs on \((X, u)\) equipped with the restriction of \(N\) to
\(\nu_q(X)\), which we shall again denote by \(N\).
Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. For each \(a \in X\), \(f_a(x) := (u(a, x), u(x, a))\) whenever \(x \in X\), is a strongly tight pair belonging to \(UT(X, u)\).

Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. We say that a function pair \(f = (f_1, f_2)\) is minimal among the strongly tight pairs on \((X, u)\) if it is a strongly tight pair and if \(g = (g_1, g_2)\) is strongly tight on \((X, u)\) and for each \(x \in X\), \(g_1(x) \leq f_1(x)\) and \(g_2(x) \leq f_2(x)\), then \(f = g\).

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In the following we shall call \((\nu_q(X), N)\) the ultra-quasi-metrically injective hull of \((X, u)\). The reason for this name will be explained later.
Lemma

Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space and let \(f \in \nu_q(X)\). For all \(x, y \in X\), \((f_1(x) > f_1(y))\) implies that \(f_1(x) \leq u(y, x)\) and \((f_2(x) > f_2(y))\) implies that \(f_2(x) \leq u(x, y)\).
Lemma

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Corollary

Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. If \(f = (f_1, f_2)\) is a minimal strongly tight function pair on \((X, u)\), then \(f_1(x) \leq \max\{f_1(y), u(y, x)\}\) and \(f_2(x) \leq \max\{f_2(y), u(x, y)\}\) whenever \(x, y \in X\). Thus \(f_1 : (X, u) \to ([0, \infty), n^{-1})\) and \(f_2 : (X, u) \to ([0, \infty), n)\) are contracting maps (see Corollary above).
Lemma

Suppose that \((f_1, f_2)\) is a minimal strongly tight pair of functions on a \(T_0\)-ultra-quasi-metric space \((X, u)\). Then

\[
f_2(x) = \sup\{u(x, y) : y \in X \text{ and } u(x, y) > f_1(y)\} = \\
\sup\{(f_2)_1(y) : y \in X \text{ and } (f_2)_1(y) > f_1(y)\}
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and

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and

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f_1(x) = \sup \{ u(y, x) : y \in X \text{ and } u(y, x) > f_2(y) \} = \sup \{ (f_x)_2(y) : y \in X \text{ and } (f_x)_2(y) > f_2(y) \}
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whenever \(x \in X\).
Lemma

Let \((f_1, f_2), (g_1, g_2)\) be minimal strongly tight pairs of functions on a \(T_0\)-ultra-quasi-metric space \((X, u)\). Then

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N((f_1, f_2), (g_1, g_2)) = \sup_{x \in X} n(f_1(x), g_1(x)) = \sup_{x \in X} n(g_2(x), f_2(x)).
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Lemma

Let \((f_1, f_2), (g_1, g_2)\) be minimal strongly tight pairs of functions on a \(T_0\)-ultra-quasi-metric space \((X, u)\). Then

\[ N((f_1, f_2), (g_1, g_2)) = \sup_{x \in X} n(f_1(x), g_1(x)) = \sup_{x \in X} n(g_2(x), f_2(x)). \]

Corollary

Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. Any minimal strongly tight function pair \(f = (f_1, f_2)\) on \(X\) satisfies the following conditions:

\[ f_1(x) = \sup_{y \in X} n(u(y, x), f_2(y)) = \sup_{y \in X} n(f_1(y), u(x, y)) \]

and

\[ f_2(x) = \sup_{y \in X} n(u(x, y), f_1(y)) = \sup_{y \in X} n(f_2(y), u(y, x)) \]

whenever \(x \in X\).
Proposition Let $f = (f_1, f_2)$ be a strongly tight function pair on a $T_0$-ultra-quasi-metric space $(X, u)$ such that

$$f_1(x) \leq \max\{f_1(y), u(y, x)\} \text{ and } f_2(x) \leq \max\{f_2(y), u(x, y)\}$$

whenever $x, y \in X$. 
**Proposition** Let $f = (f_1, f_2)$ be a strongly tight function pair on a $T_0$-ultra-quasi-metric space $(X, u)$ such that
\[
f_1(x) \leq \max\{f_1(y), u(y, x)\} \text{ and } f_2(x) \leq \max\{f_2(y), u(x, y)\}
\]
whenever $x, y \in X$. Furthermore suppose that there is a sequence $(a_n)_{n \in \mathbb{N}}$ in $X$ with $\lim_{n \to \infty} f_1(a_n) = 0$ and $\lim_{n \to \infty} f_2(a_n) = 0$. Then $f$ is a minimal strongly tight pair.
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Lemma

Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. For each \(a \in X\), the pair \(f_a\) belongs to \(\nu_q(X, u)\).
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Theorem

Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. For each \(f \in \nu_q(X, u)\) and \(a \in X\) we have that \(N(f, f_a) = f_1(a)\) and \(N(f_a, f) = f_2(a)\). The map \(e_X : (X, u) \rightarrow (\nu_q(X, u), N)\) defined by \(e_X(a) = f_a\) whenever \(a \in X\) is an isometric embedding.
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Corollary

Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. Then \(N\) is indeed a \(T_0\)-ultra-quasi-metric on \(\nu_q(X)\).
Lemma

Suppose that $(X, u)$ is a $T_0$-ultra-quasi-metric space and $(f_1, f_2) \in \nu_q(X, u)$ such that $f_1(a) = 0 = f_2(a)$ for some $a \in X$. Then $(f_1, f_2) = e_X(a)$. 
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Suppose that \((X, u)\) is a \(T_0\)-ultra-quasi-metric space and \((f_1, f_2) \in \nu_q(X, u)\) such that \(f_1(a) = 0 = f_2(a)\) for some \(a \in X\). Then \((f_1, f_2) = e_X(a)\).

Lemma

Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. Then for any \(f, g \in \nu_q(X, u)\) we have that \(N(f, g) = \sup\{u(x_1, x_2) : x_1, x_2 \in X, u(x_1, x_2) > f_2(x_1) \text{ and } u(x_1, x_2) > g_1(x_2)\}\).
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Let \((X, u)\) be an ultra-quasi-pseudometric space and for each \(x \in X\) and \(r \in [0, \infty)\) let \(C_u(x, r) = \{y \in X : u(x, y) \leq r\}\) be the \(\tau(u^{-1})\)-closed ball of radius \(r\) at \(x\).
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**Lemma**

*Let \((X, u)\) be an ultra-quasi-pseudometric space. Moreover let \(x, y \in X\) and \(r, s \geq 0\). Then \(C_u(x, r) \cap C_{u^{-1}}(y, s) \neq \emptyset\) if and only if \(u(x, y) \leq \max\{r, s\}\).*
Let \((X, u)\) be an ultra-quasi-pseudometric space and for each \(x \in X\) and \(r \in [0, \infty)\) let \(C_u(x, r) = \{y \in X : u(x, y) \leq r\}\) be the \(\tau(u^{-1})\)-closed ball of radius \(r\) at \(x\).

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**Definition**

Let \((X, u)\) be an ultra-quasi-pseudometric space. Let \((x_i)_{i \in I}\) be a family of points in \(X\) and let \((r_i)_{i \in I}\) and \((s_i)_{i \in I}\) be families of non-negative reals.
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Let \((X, u)\) be an ultra-quasi-pseudometric space and for each \(x \in X\) and \(r \in [0, \infty)\) let \(C_u(x, r) = \{y \in X : u(x, y) \leq r\}\) be the \(\tau(u^{-1})\)-closed ball of radius \(r\) at \(x\).

**Lemma**

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Example

The $T_0$-ultra-quasi-metric space $([0, \infty), n)$ is $q$-spherically complete.
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Proposition (a) Let $(X, u)$ be an ultra-quasi-pseudometric space. Then $(X, u)$ is $q$-spherically complete if and only if $(X, u^{-1})$ is $q$-spherically complete.
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(a) Let $(X, u)$ be an ultra-quasi-pseudometric space. Then $(X, u)$ is $q$-spherically complete if and only if $(X, u^{-1})$ is $q$-spherically complete.

(b) Let $(X, u)$ be a $T_0$-ultra-quasi-metric space. If $(X, u)$ is $q$-spherically complete, then $(X, u^s)$ is spherically complete.
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Proposition Each $q$-spherically complete $T_0$-ultra-quasi-metric space $(X, u)$ is bicomplete.
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Proposition

Each $q$-spherically complete $T_0$-ultra-quasi-metric space $(X, u)$ is bicomplete.

Theorem

A $T_0$-ultra-quasi-metric space is $q$-spherically complete if and only if it is ultra-quasi-metrically injective.
Proposition Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. Then \((f_1, f_2) \in \nu_q(X, u)\) implies that \((f_2, f_1) \in \nu_q(X, u^{-1})\). Therefore

\[ s : (\nu_q(X, u), N) \rightarrow (\nu_q(X, u^{-1}), N^{-1}) \]

where \(s\) is defined by \(s((f, g)) = (g, f)\) whenever \((f, g) \in \nu_q(X, u)\) is a bijective isometric map. (Indeed the ultra-quasi-metrically injective hull \((\nu_q(X, u), N)\) of \((X, u)\) is isometric to the conjugate space of the ultra-quasi-metrically injective hull \((\nu_q(X, u^{-1}), N)\) of \((X, u^{-1})\).)
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Proposition Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space. If \(s = (s_1, s_2)\) is a minimal strongly tight pair of functions on the \(T_0\)-ultra-quasi-metric space \((\nu_q(X), N)\), then \(s \circ e_X\) is a minimal strongly tight pair of functions on \((X, u)\).
**Proposition** The following statements are true for any $T_0$-ultra-quasi-metric space $(X, u)$.

(a) $(\nu_q(X), N)$ is $q$-spherically complete.
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(a) $(\nu_q(X), N)$ is $q$-spherically complete.
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Corollary

The following statements are equivalent for a $T_0$-ultra-quasi-metric space $(X, u)$:

(a) $(X, u)$ is $q$-spherically complete.
(b) For each $f \in \nu_q(X)$ there is $x \in X$ such that $f_1 = (f_x)_1$ and $f_2 = (f_x)_2$.
(c) For each $f \in \nu_q(X)$ there is $x \in X$ such that $f_1(x) = 0 = f_2(x)$. 
Recall that a quasi-pseudometric space \((X, d)\) is called \textit{totally bounded} provided that the pseudometric space \((X, d^s)\) is totally bounded.
Recall that a quasi-pseudometric space $(X, d)$ is called \textit{totally bounded} provided that the pseudometric space $(X, d^s)$ is totally bounded.

\textbf{Lemma}

\textit{Let $(X, u)$ be a $T_0$-ultra-quasi-metric space that is totally bounded and let $\epsilon > 0$. Then there is a finite subset $E$ of $X$ such that}

\[ \{f_1(x) : f \in \nu_q(X), x \in X, f_1(x) > \epsilon\} \cup \{f_2(x) : f \in \nu_q(X), x \in X, f_2(x) > \epsilon\} = \{u(e, e') : e, e' \in E, u(e, e') > \epsilon\}. \]

\textit{In particular, there is a real $b > 0$ such that for any $f = (f_1, f_2) \in \nu_q(X)$ we have $(f_1(X) \cup f_2(X)) \subseteq [0, b]$.}
Recall that a quasi-pseudometric space \((X, d)\) is called *totally bounded* provided that the pseudometric space \((X, d^s)\) is totally bounded.

**Lemma**

Let \((X, u)\) be a \(T_0\)-ultra-quasi-metric space that is totally bounded and let \(\epsilon > 0\). Then there is a finite subset \(E\) of \(X\) such that

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**Proposition** If \((X, u)\) is a totally bounded \(T_0\)-ultra-quasi-metric space, then the \(T_0\)-ultra-quasi-metric space \((\nu_q(X, u), N)\) is totally bounded, too.
Recall that a quasi-pseudometric space \((X, d)\) is called *totally bounded* provided that the pseudometric space \((X, d^s)\) is totally bounded.

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**Corollary**

Let \((X, m)\) be a totally bounded ultra-metric space. Then the completion of \((X, m)\) is isometric to \((\nu_s(X), E)\).
Example

Let $X = \{0, 1\}$ be equipped with the discrete metric $u$ defined by $u(x, y) = 1$ if $x \neq y$, and $u(x, y) = 0$ otherwise. Then $(X, u)$ is not $q$-spherically complete, although it is spherically complete.
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Thank you for attention

Merci