The ultra-quasi-metric injective hull of a T_0-ultra-quasi-metric space

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- Recently Kemajou, Kunzi and Otafudu have developed a concept of hyperconvexity (called Isbell-convexity) that is appropriate in the category of T_0-quasi-metric spaces and non-expansive maps.In particular an explicit construction of the corresponding hull (called Isbell-hull) was provided.

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(iii) for any $x, y \in X$, $u(x, y) = 0 = u(y, x)$ implies that $x = y$, then u is called a *T_0-ultra-quasi-metric*.

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Observe that then $u^s = u \vee u^{-1}$ is an *ultra-metric* on X.

Example

Let $X = [0, \infty)$ be equipped with $n(x, y) = x$ if $x, y \in X$ and $x > y$, and $n(x, y) = 0$ if $x, y \in X$ and $x \leq y$. It is easy to check that (X, n) is a T_0-ultra-quasi-metric space.

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Note also that for $x, y \in [0, \infty)$ we have $n^s(x, y) = \max\{x, y\}$ if $x \neq y$ and $n(x, y) = 0$ if $x = y$. Observe that the ultra-metric n^s is complete on $[0, \infty)$.

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Lemma

Let $a, b, c \in [0, \infty)$. Then the following conditions are equivalent:
(a) $n(a, b) \leq c$.
(b) $a \leq \max\{b, c\}$.

Corollary

Let (X, u) *be an ultra-quasi-pseudometric space. Consider a map* $f : X \to [0, \infty)$ *and let* $x, y \in X$*. Then the following are equivalent:*
(a) $n(f(x), f(y)) \leq u(x, y)$;
(b) $f(x) \leq \max\{f(y), u(x, y)\}$.

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Corollary

Let (X, u) be an ultra-quasi-pseudometric space.
(a) Then $f : (X, u) \to ([0, \infty), n)$ *is a contracting map if and only if* $f(x) \leq \max\{f(y), u(x, y)\}$ *whenever* $x, y \in X$.
(b) Then $f : (X, u) \to ([0, \infty), n^{-1})$ *is a contracting map if and only if* $f(x) \leq \max\{f(y), u(y, x)\}$ *whenever* $x, y \in X$.

Definition

Let (X, u) be a T_0-ultra-quasi-metric space and let $\mathcal{FP}(X, u)$ be the set of all pairs $f = (f_1, f_2)$ of functions where $f_i : X \to [0, \infty)$ $(i = 1, 2)$.

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For any such pairs (f_1,f_2) and (g_1,g_2) set

$$N((f_1,f_2),(g_1,g_2)) = \max\{\sup_{x\in X} n(f_1(x),g_1(x)), \sup_{x\in X} n(g_2(x),f_2(x))\}.$$

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It is obvious that N is an extended T_0-ultra-quasi-metric on the set $\mathcal{FP}(X, u)$ of these function pairs.

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Let (X, u) be a T_0-ultra-quasi-metric space. We shall say that a pair $f \in \mathcal{FP}(X, u)$ is *strongly tight* if for all $x, y \in X$, we have $u(x, y) \leq \max\{f_2(x), f_1(y)\}$.

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The set of all strongly tight function pairs of a T_0-ultra-quasi-metric space (X, u) will be denoted by $\mathcal{UT}(X, u)$.

Lemma

Let (X, u) be a T_0-ultra-quasi-metric space. For each $a \in X$, $f_a(x) := (u(a, x), u(x, a))$ whenever $x \in X$, is a strongly tight pair belonging to $\mathcal{UT}(X, u)$.

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Let (X, u) be a T_0-ultra-quasi-metric space. We say that a function pair $f = (f_1, f_2)$ is *minimal* among the strongly tight pairs on (X, u) if it is a strongly tight pair and if $g = (g_1, g_2)$ is strongly tight on (X, u) and for each $x \in X$, $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$, then $f = g$.

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By $\nu_q(X, u)$ (or more briefly, $\nu_q(X)$) we shall denote the set of all minimal strongly tight function pairs on (X, u) equipped with the restriction of N to $\nu_q(X)$, which we shall again denote by N.

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In the following we shall call $(\nu_q(X), N)$ the *ultra-quasi-metrically injective hull* of (X, u). The reason for this name will be explained later.

Lemma

Let (X, u) *be a* T_0*-ultra-quasi-metric space and let* $f \in \nu_q(X)$*. For all* $x, y \in X$*,* $(f_1(x) > f_1(y)$ *implies that* $f_1(x) \leq u(y, x))$ *and* $(f_2(x) > f_2(y)$ *implies that* $f_2(x) \leq u(x, y))$.

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Corollary

Let (X, u) be a T_0-ultra-quasi-metric space. If $f = (f_1, f_2)$ is a minimal strongly tight function pair on (X, u), then $f_1(x) \leq \max\{f_1(y), u(y, x)\}$ and $f_2(x) \leq \max\{f_2(y), u(x, y)\}$ whenever $x, y \in X$. Thus $f_1 : (X, u) \to ([0, \infty), n^{-1})$ and $f_2 : (X, u) \to ([0, \infty), n)$ are contracting maps (see Corollary above).

Lemma

Suppose that (f_1, f_2) is a minimal strongly tight pair of functions on a T_0-ultra-quasi-metric space (X, u). Then

$$f_2(x) = \sup\{u(x,y) : y \in X \textit{ and } u(x,y) > f_1(y)\} =$$

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Corollary

Let (X, u) be a T_0-ultra-quasi-metric space. Any minimal strongly tight function pair $f = (f_1, f_2)$ on X satisfies the following conditions:

$$f_1(x) = \sup_{y \in X} n(u(y, x), f_2(y)) = \sup_{y \in X} n(f_1(y), u(x, y))$$

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Proposition Let $f = (f_1, f_2)$ be a strongly tight function pair on a T_0-ultra-quasi-metric space (X, u) such that

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whenever $x, y \in X$.Furthermore suppose that there is a sequence $(a_n)_{n\in\mathbb{N}}$ in X with $\lim_{n\to\infty} f_1(a_n) = 0$ and $\lim_{n\to\infty} f_2(a_n) = 0$. Then f is a minimal strongly tight pair.

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Theorem

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Theorem

Let (X, u) be a T_0-ultra-quasi-metric space. For each $f \in \nu_q(X, u)$ and $a \in X$ we have that $N(f, f_a) = f_1(a)$ and $N(f_a, f) = f_2(a)$. The map $e_X : (X, u) \to (\nu_q(X, u), N)$ defined by $e_X(a) = f_a$ whenever $a \in X$ is an isometric embedding.

Corollary

Let (X, u) be a T_0-ultra-quasi-metric space. Then N is indeed a T_0-ultra-quasi-metric on $\nu_q(X)$.

Lemma

Suppose that (X, u) is a T_0-ultra-quasi-metric space and $(f_1, f_2) \in \nu_q(X, u)$ such that $f_1(a) = 0 = f_2(a)$ for some $a \in X$. Then $(f_1, f_2) = e_X(a)$.

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Lemma

Let (X, u) be a T_0-ultra-quasi-metric space. Then for any $f, g \in \nu_q(X, u)$ we have that $N(f, g) = \sup\{u(x_1, x_2) : x_1, x_2 \in X,\ u(x_1, x_2) > f_2(x_1)$ and $u(x_1, x_2) > g_1(x_2)\}$.

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Suppose that (X, u) *is a* T_0*-ultra-quasi-metric space and* $(f_1, f_2) \in \nu_q(X, u)$ *such that* $f_1(a) = 0 = f_2(a)$ *for some* $a \in X$*. Then* $(f_1, f_2) = e_X(a)$*.*

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Let (X, u) be an ultra-quasi-pseudometric space and for each $x \in X$ and $r \in [0, \infty)$ let $C_u(x, r) = \{y \in X : u(x, y) \leq r\}$ be the $\tau(u^{-1})$-closed ball of radius r at x.

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Lemma

Let (X, u) be an ultra-quasi-pseudometric space. Moreover let $x, y \in X$ and $r, s \geq 0$. Then $C_u(x, r) \cap C_{u^{-1}}(y, s) \neq \emptyset$ if and only if $u(x, y) \leq \max\{r, s\}$.

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Definition

Let (X, u) be an ultra-quasi-pseudometric space. Let $(x_i)_{i \in I}$ be a family of points in X and let $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ be families of non-negative reals.

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Example

The T_0-ultra-quasi-metric space $([0, \infty), n)$ is q-spherically complete.

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Proposition Each q-spherically complete T_0-ultra-quasi-metric space (X,u) is bicomplete.

Theorem

A T_0-ultra-quasi-metric space is q-spherically complete if and only if it is ultra-quasi-metrically injective.

Proposition Let (X, u) be a T_0-ultra-quasi-metric space. Then $(f_1, f_2) \in \nu_q(X, u)$ implies that $(f_2, f_1) \in \nu_q(X, u^{-1})$. Therefore

$$s : (\nu_q(X, u), N) \rightarrow (\nu_q(X, u^{-1}), N^{-1})$$

where s is defined by $s((f, g)) = (g, f)$ whenever $(f, g) \in \nu_q(X, u)$ is a bijective isometric map. (Indeed the ultra-quasi-metrically injective hull $(\nu_q(X, u), N)$ of (X, u) is isometric to the conjugate space of the ultra-quasi-metrically injective hull $(\nu_q(X, u^{-1}), N)$ of (X, u^{-1}).)

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Proposition Let (X, u) be a T_0-ultra-quasi-metric space. If $s = (s_1, s_2)$ is a minimal strongly tight pair of functions on the T_0-ultra-quasi-metric space $(\nu_q(X), N)$, then $s \circ e_X$ is a minimal strongly tight pair of functions on (X, u).

Proposition The following statements are true for any T_0-ultra-quasi-metric space (X, u).
(a) $(\nu_q(X), N)$ is q-spherically complete.

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(a) $(\nu_q(X), N)$ is q-spherically complete.
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Corollary

The following statements are equivalent for a T_0-ultra-quasi-metric space (X, u) :
(a) (X, u) is q-spherically complete.
(b) For each $f \in \nu_q(X)$ there is $x \in X$ such that $f_1 = (f_x)_1$ and $f_2 = (f_x)_2$.
(c) For each $f \in \nu_q(X)$ there is $x \in X$ such that $f_1(x) = 0 = f_2(x)$.

Recall that a quasi-pseudometric space (X, d) is called *totally bounded* provided that the pseudometric space (X, d^s) is totally bounded.

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Lemma

Let (X, u) be a T_0-ultra-quasi-metric space that is totally bounded and let $\epsilon > 0$. Then there is a finite subset E of X such that

$$\{f_1(x) : f \in \nu_q(X), x \in X, f_1(x) > \epsilon\} \cup \{f_2(x) : f \in \nu_q(X), x \in X, f_2(x) > \epsilon\} =$$

$$\{u(e, e') : e, e' \in E, u(e, e') > \epsilon\}.$$

In particular, there is a real $b > 0$ such that for any $f = (f_1, f_2) \in \nu_q(X)$ we have $(f_1(X) \cup f_2(X)) \subseteq [0, b]$.

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Proposition If (X, u) is a totally bounded T_0-ultra-quasi-metric space, then the T_0-ultra-quasi-metric space $(\nu_q(X, u), N)$ is totally bounded, too.

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Corollary

Let (X, m) be a totally bounded ultra-metric space. Then the completion of (X, m) is isometric to $(\nu_s(X), E)$.

Example

Let $X = \{0, 1\}$ be equipped with the discrete metric u defined by $u(x, y) = 1$ if $x \neq y$, and $u(x, y) = 0$ otherwise. Then (X, u) is not q-spherically complete, although it is spherically complete.

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Thank you for attention
Merci