INDUCTION EXERCISES 1

1. Factorials are defined inductively by the rule

$$0! = 1 \quad \text{and} \quad (n+1)! = n! \times (n+1).$$

Then binomial coefficients are defined for $0 \leq k \leq n$ by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Prove *from these definitions* that

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1},$$

and deduce the Binomial Theorem: that for any x and y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

2. Prove that

$$\sum_{r=1}^{n} \frac{1}{r^2} \leq 2 - \frac{1}{n}.$$

3. Prove that for $n = 1, 2, 3, \ldots$

$$\sqrt{n} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1.$$

4. Let $A = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}$. Show that

$$A^n = 3^{n-1} \begin{pmatrix} 2n+3 & -n \\ 4n & 3-2n \end{pmatrix}$$

for $n = 1, 2, 3, \ldots$ Can you find a matrix B such that $B^2 = A$?

5. Let k be a positive integer. Prove by induction on n that

$$\sum_{r=1}^{n} r(r+1)(r+2)\cdots(r+k-1) = \frac{n(n+1)(n+2)\cdots(n+k)}{k+1}.$$

Show now by induction on k that

$$\sum_{r=1}^{n} r^k = \frac{n^{k+1}}{k+1} + E_k(n)$$

where $E_k(n)$ is a polynomial in n of degree at most k.

INDUCTION EXERCISES 2.

1. Show that n lines in the plane, no two of which are parallel and no three meeting in a point, divide the plane into

$$\frac{n^2+n+2}{2}$$

regions.

2. Prove for every positive integer n, that

$$3^{3n-2}+2^{3n+1}$$

is divisible by 19.

3. (a) Show that if $u^2-2v^2=1$ then

$$(3u+4v)^2-2(2u+3v)^2=1.$$

(b) Beginning with $u_0=3, v_0=2$, show that the recursion

$$u_{n+1}=3u_n+4v_n \quad \text{and} \quad v_{n+1}=2u_n+3v_n$$

generates infinitely many integer pairs (u,v) which satisfy $u^2-2v^2=1$.

(c) How can this process be used to produce better and better rational approximations to $\sqrt{2}$? How many times need this process be repeated to produce a rational approximation accurate to 6 decimal places?

4. The Fibonacci numbers F_n are defined by the recurrence relation

$$F_n=F_{n-1}+F_{n-2}, \quad \text{for} \quad n\geq 2$$

and $F_0=0$ and $F_1=1$. Prove for every integer $n\geq 0$, that

$$F_n=\frac{\alpha^n-\beta^n}{\sqrt{5}}$$

where

$$\alpha=\frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta=\frac{1-\sqrt{5}}{2}.$$

[Hint: you may find it helpful to show first that the two roots of the equation $x^2=x+1$ are α and β.]

5. The sequence of numbers $x_0, x_1, x_2, \ldots$ begins with $x_0=1$ and $x_1=1$ and is then recursively determined by the equations

$$x_{n+2}=4x_{n+1}-3x_n+3^n \quad \text{for} \quad n\geq 0.$$

(a) Find the values of x_2, x_3, x_4 and x_5.

(b) Can you find a solution of the form

$$x_n=A+B\times 3^n+C\times n3^n$$

which agrees with the values of $x_0,\ldots,x_5$ that you have found?

(c) Use induction to prove that this is the correct formula for x_n for all $n\geq 0$.

ALGEBRA EXERCISES 1

1. (a) Find the remainder when $n^2 + 4$ is divided by 7 for $0 \leq n < 7$.

Deduce that $n^2 + 4$ is not divisible by 7, for every positive integer n. [Hint: write $n = 7k + r$ where $0 \leq r < 7$.]

(b) Now k is an integer such that $n^3 + k$ is not divisible by 4 for all integers n. What are the possible values of k?

2. (i) Prove that if a, b are positive real numbers then

$$\sqrt{ab} \leq \frac{1}{2}(a+b).$$

(ii) Now let $a_1, a_2, \ldots, a_n$ be positive real numbers. Let $S = a_1 + a_2 + \cdots + a_n$ and $P = a_1 a_2 \cdots a_n$.

Suppose that a_i and a_j are distinct. Show that replacing a_i and a_j with $(a_i + a_j)/2$ and $(a_i + a_j)/2$ increases P without changing S.

Deduce that

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

3. (i) Let n be a positive integer. Show that

$$x^n - y^n = (x-y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right).$$

(ii) Let a also be a positive integer. Show that if $a^n - 1$ is prime then $a = 2$ and n is prime.

Is it true that if n is prime then $2^n - 1$ is also prime?

4. Let a, b, r, s be rational numbers with $s \neq 0$. Suppose that the number $r + s\sqrt{2}$ is a root of the quadratic equation

$$x^2 + ax + b = 0.$$

Show that $r - s\sqrt{2}$ is also a root.

5. (i) The cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots α, β, γ, and so factorises as

$$a(x-\alpha)(x-\beta)(x-\gamma).$$

Determine

$$\alpha + \beta + \gamma, \quad \alpha\beta + \beta\gamma + \gamma\alpha, \quad \alpha\beta\gamma,$$

in terms of a, b, c, d. What does $\alpha^2 + \beta^2 + \gamma^2$ equal?

(ii) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(iii) By considering the roots of the equation $4x^3 - 3x - \cos 3\theta = 0$ deduce that

$$\cos\theta\cos(\theta + 2\pi/3)\cos(\theta + 4\pi/3) = \frac{\cos(3\theta)}{4}.$$

What do

$$\cos\theta + \cos(\theta + 2\pi/3) + \cos(\theta + 4\pi/3) \quad \text{and} \quad \cos^2\theta + \cos^2(\theta + 2\pi/3) + \cos^2(\theta + 4\pi/3)$$

equal?

ALGEBRA EXERCISES 2

1. Under what conditions on the real numbers a, b, c, d, e, f do the simultaneous equations

$$ax + by = e \quad \text{and} \quad cx + dy = f$$

have (a) a unique solution, (b) no solution, (c) infinitely many solutions in x and y.

Select values of a, b, c, d, e, f for each of these cases, and sketch on separate axes the lines $ax+by = e$ and $cx+dy = f$.

2. For what values of a do the simultaneous equations

$$\begin{aligned} x + 2y + a^2 z &= 0, \\ x + ay + z &= 0, \\ x + ay + a^2 z &= 0, \end{aligned}$$

have a solution other than $x = y = z = 0$. For each such a find the general solution of the above equations.

3. Do 2×2 matrices exist satisfying the following properties? Either find such matrices or show that no such exist.
(i) A such that $A^5 = I$ and $A^i \neq I$ for $1 \leq i \leq 4$,
(ii) A such that $A^n \neq I$ for all positive integers n,
(iii) A and B such that $AB \neq BA$,
(iv) A and B such that AB is invertible and BA is singular (i.e. not invertible),
(v) A such that $A^5 = I$ and $A^{11} = 0$.

4. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and let} \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

be a 2×2 matrix and its *transpose.* Suppose that $\det A = 1$ and

$$A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $a^2 + c^2 = 1$, and hence that a and c can be written as

$$a = \cos\theta \quad \text{and} \quad c = \sin\theta.$$

for some θ in the range $0 \leq \theta < 2\pi$. Deduce that A has the form

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

5. (a) Prove that

$$\det(AB) = \det(A)\det(B)$$

for any 2×2 matrices A and B.

(b) Let A denote the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Show that

$$A^2 - (\text{trace}A)A + (\det A)I = 0 \qquad (1)$$

where

- $\text{trace}A = a + d$ is the trace of A, that is the sum of the diagonal elements;
- $\det A = ad - bc$ is the determinant of A;
- I is the 2×2 identity matrix.

(c) Suppose now that $A^n = 0$ for some $n \geq 2$. Prove that $\det A = 0$. Deduce using equation (1) that $A^2 = 0$.

CALCULUS EXERCISES 1 — Curve Sketching

1. Sketch the graph of the curve

$$y = \frac{x^2+1}{(x-1)\,(x-2)}$$

carefully labelling any turning points and asymptotes.

2. The parabola $x = y^2 + ay + b$ crosses the parabola $y = x^2$ at $(1, 1)$ making right angles.

Calculate the values of a and b.

On the same axes, sketch the two parabolas.

3. The curve C in the xy-plane has equation

$$x^2 + xy + y^2 = 1.$$

By solving $\mathrm{d}y/\mathrm{d}x = 0$, show that the maximum and minimum values taken by y are $\pm 2/\sqrt{3}$.

By changing to polar co-ordinates, ($x = r\cos\theta$, $y = r\sin\theta$),sketch the curve C.

What is the greatest distance of a point on C from the origin?

4. Sketch the curve $y = x^3 + ax + b$ for a selection of values of a and b.

Suppose now that a is negative. Find the co-ordinates of the turning points of the graph and deduce that $y = 0$ has exactly two roots when

$$b = \pm\frac{2a}{3}\sqrt{\frac{-a}{3}}$$

For what values of b does the equation $y = 0$ have three distinct real roots?

5. On separate xu- and yu-axes sketch the curves $u = 8(x^3 - x)$ and $u = e^y/y$ labelling all turning points.

[Harder] Hence sketch the curve $e^y = 8y(x^3 - x)$.

CALCULUS EXERCISES 2 — Numerical Methods and Estimation

1. Use calculus, or trigonometric identities, to prove the following inequalities for θ in the range $0 < \theta < \frac{\pi}{2}$:

- $\sin\theta < \theta$;
- $\theta < \tan\theta$;
- $\cos 2\theta < \cos^2\theta$.

Hence, without directly calculating the following integrals, rank them in order of size.

$$\text{(a)} \int_0^1 x^3 \cos x \, \mathrm{d}x, \qquad \text{(b)} \int_0^1 x^3 \cos^2 x \, \mathrm{d}x, \qquad \text{(c)} \int_0^1 x^2 \sin x \cos x \, \mathrm{d}x, \qquad \text{(d)} \int_0^1 x^3 \cos 2x \, \mathrm{d}x.$$

2. Show that the equation

$$\sin x = \frac{1}{2}x$$

has three roots. Using Newton-Raphson, or a similar numerical method, find the positive root to 6 d.p.

The equation $\sin x = \lambda x$ has three real roots when $\lambda = \alpha$ or when $\beta < \lambda < 1$ for two real numbers $\alpha < 0 < \beta$. Plot, on the same axes, the curves

$$y = \sin x, \qquad y = \alpha x, \qquad y = \beta x.$$

3. Let S denote the circle in the xy-plane with centre (0,0) and radius 1. A regular m-sided polygon I_m is inscribed in S and a regular n-sided polygon C_n is circumscribed about S.

(a) By considering the perimeter of I_m and the area bounded by C_n, prove that:

$$m \sin\left(\frac{\pi}{m}\right) < \pi < n \tan\left(\frac{\pi}{n}\right),$$

for all natural numbers $m, n \geq 3$.

(b) Archimedes showed (using this method) that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. What are the smallest values of m and n needed to verify Archimedes' inequality?

4. Find the coefficients of $1, x, x^2, x^3, x^4$ in the power series expansion (Taylor's series expansion) for $f(x) = \sec x$.

Use this approximation to make an estimate for $\sec\frac{1}{10}$. With the aid of a calculator, find to how many decimal places the approximation is accurate.

5. Show that $\int \ln x \, \mathrm{d}x = x\ln x - x + \text{constant}$

Sketch the graph of the equation $y = \ln x$. By consideration of areas on your graph, show that

$$n \ln n - n + 1 < \sum_1^n \ln r < (n+1)\ln(n+1) - n.$$

Let $G_n = \sqrt[n]{n!}$ denote the geometric mean of 1, 2, ..., n. Show that G_n/n approaches $1/e$ as n becomes large

CALCULUS EXERCISES 3 — Techniques of Integration

1. Evaluate

$$\int \frac{\ln x}{x}\ \mathrm{d}x, \qquad \int x \sec^2 x\ \mathrm{d}x, \qquad \int_3^\infty \frac{\mathrm{d}x}{(x-1)(x-2)}, \qquad \int_0^1 \tan^{-1} x\ \mathrm{d}x, \qquad \int_0^1 \frac{\mathrm{d}x}{e^x+1}.$$

2. Evaluate, using trigonometric and/or hyperbolic substitutions,

$$\int \frac{\mathrm{d}x}{x^2+1}, \qquad \int_1^2 \frac{\mathrm{d}x}{\sqrt{x^2-1}}, \qquad \int \frac{\mathrm{d}x}{\sqrt{4-x^2}}, \qquad \int_2^\infty \frac{\mathrm{d}x}{(x^2-1)^{3/2}}$$

3. By completing the square in the denominator, and using the substitution

$$x = \frac{\sqrt{2}}{3}\tan\theta - \frac{1}{3}$$

evaluate

$$\int \frac{\mathrm{d}x}{3x^2+2x+1}..$$

By similarly completing the square in the following denominators, and making appropriate trigonometric and/or hyperbolic substitutions, evaluate the following integrals

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+2x+5}}, \qquad \int_0^\infty \frac{\mathrm{d}x}{4x^2+4x+5}.$$

4. Let $t = \tan\frac{1}{2}\theta$. Show that

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}, \qquad \tan\theta = \frac{2t}{1-t^2}$$

and that

$$\mathrm{d}\theta = \frac{2\ \mathrm{d}t}{1+t^2}.$$

Use the substitution $t = \tan\frac{1}{2}\theta$ to evaluate

$$\int_0^{\pi/2} \frac{\mathrm{d}\theta}{(1+\sin\theta)^2}.$$

5. Let

$$I_n = \int_0^{\pi/2} x^n \sin x\ \mathrm{d}x.$$

Evaluate I_0 and I_1.

Show, using integration by parts, that

$$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n\,(n-1)\,I_{n-2}.$$

Hence, evaluate I_5 and I_6.

CALCULUS EXERCISES 4 — Differential Equations

1. Find the general solutions of the following separable differential equations.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos^2 x}{\cos^2 2y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = e^{x+2y}.$$

2. Find the solution of the following initial value problems. On separate axes sketch the solution to each problem.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1-2x}{y}, \qquad y(1) = -2, \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x\left(x^2+1\right)}{4y^3}, \qquad y(0) = \frac{-1}{\sqrt{2}}, \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1+y^2}{1+x^2} \text{ where } y(0) = 1. \end{aligned}$$

3. The equation for *Simple Harmonic Motion,* with constant frequency ω, is

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x.$$

Show that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

where $v = \mathrm{d}x/\mathrm{d}t$ denotes velocity. Find and solve a separable differential equation in v and x given that $x = a$ when $v = 0$.

Hence show that

$$x(t) = a\sin(\omega t + \varepsilon)$$

for some constant ε.

4. Find the most general solution of the following homogeneous constant coefficient differential equations:

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - y &= 0, \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y &= 0, \quad \text{where } y(0) = y'(0) = 1, \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y &= 0, \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y &= 0, \quad \text{where } y(0) = y'(0) = 1. \end{aligned}$$

5. Write the left hand side of the differential equation

$$(2x+y) + (x+2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x,y)) = 0,$$

where $F(x,y)$ is a polynomial in x and y. Hence find the general solution of the equation.

Use this method to find the general solution of

$$(y\cos x + 2xe^y) + \left(\sin x + x^2e^y - 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

CALCULUS EXERCISES 5 — Further Differential Equations

1. Find all solutions of the following separable differential equations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-xy}{xy-x},$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin^{-1}x}{y^2\sqrt{1-x^2}}, \qquad y(0)=0.$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(1+3x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \quad \text{where } y(1)=0 \text{ and } y'(1)=\frac{-1}{2}.$$

2. Use the method of integrating factors to solve the following equations with initial conditions

$$\frac{\mathrm{d}y}{\mathrm{d}x}+xy = x \quad \text{where} \quad y(0)=0,$$
$$2x^3\frac{\mathrm{d}y}{\mathrm{d}x}-3x^2y = 1 \quad \text{where} \quad y(1)=0,$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}-y\tan x = 1 \quad \text{where} \quad y(0)=1.$$

3. Find the most general solution of the following inhomogeneous constant coefficient differential equations:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\frac{\mathrm{d}y}{\mathrm{d}x}+2y = x,$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\frac{\mathrm{d}y}{\mathrm{d}x}+2y = \sin x,$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\frac{\mathrm{d}y}{\mathrm{d}x}+2y = e^x,$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\frac{\mathrm{d}y}{\mathrm{d}x}+2y = e^{-x}.$$

4. (a) By making the substitution $y(x)=xv(x)$ in the following *homogeneous polar equations*, convert them into separable differential equations involving v and x, which you should then solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2+y^2}{xy},$$
$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y+\sqrt{x^2+y^2}.$$

(b) Make substitutions of the form $x=X+a$, $y=Y+b$, to turn the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x+y-3}{x-y-1}$$

into a homogeneous polar differential equation in X and Y. Hence find the general solution of the above equation.

5. A particle P moves in the xy-plane. Its co-ordinates $x(t)$ and $y(t)$ satisfy the equations

$$\frac{\mathrm{d}y}{\mathrm{d}t}=x+y \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t}=x-y,$$

and at time $t=0$ the particle is at $(1,0)$. Find, and solve, a homogeneous polar equation relating x and y.

By changing to polar co-ordinates $\left(r^2=x^2+y^2,\ \tan\theta=y/x\right)$, sketch the particle's journey for $t\geq 0$.

COMPLEX NUMBERS EXERCISES

1. By writing $\omega = a + ib$ (where a and b are real), solve the equation

$$\omega^2 = -5 - 12i.$$

Hence find the two roots of the quadratic equation

$$z^2 - (4+i)\,z + (5+5i) = 0.$$

2. By substituting $z = x + iy$ or $z = re^{i\theta}$ into the following equations and inequalities, sketch the following regions of the complex plane on separate Argand diagrams:

- $|z - 3 - 4i| < 5$,
- $\arg(z) = \pi/3$
- $0 \leq \mathrm{Re}((iz+3)/2) < 2$,
- $e^z = 1$,
- $\mathrm{Im}(z^2) < 0$.

3. Find the image of the point $z = 2 + it$ under each of the following transformations.

- $z \mapsto iz$,
- $z \mapsto z^2$,
- $z \mapsto e^z$,
- $z \mapsto 1/z$.

By letting t vary over all real values find the image of the line $\mathrm{Re}\, z = 2$ under the same transformations.

4. (a) Given that $e^{i\theta} = \cos\theta + i\sin\theta$, prove that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

(b) Use *De Moivre's Theorem* to show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

5. (a) Let $z = \cos\theta + i\sin\theta$ and let n be an integer. Show that

$$2\cos\theta = z + \frac{1}{z} \quad \text{and that} \quad 2i\sin\theta = z - \frac{1}{z}.$$

Find expressions for $\cos n\theta$ and $\sin n\theta$ in terms of z.

(b) Show that

$$\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$$

and hence find $\int_0^{\pi/2} \cos^5\theta \; \mathrm{d}\theta$.

GEOMETRY EXERCISES

1. Describe the regions of space given by the following vector equations. In each, $\mathbf{r}$ denotes the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; '$\cdot$' and $\wedge$ denote the scalar (dot) and vector (cross) product:

- $\mathbf{r} \wedge (\mathbf{i} + \mathbf{j}) = (\mathbf{i} - \mathbf{j})$,
- $\mathbf{r} \cdot \mathbf{i} = 1$,
- $|\mathbf{r} - \mathbf{i}| = |\mathbf{r} - \mathbf{j}|$,
- $|\mathbf{r} - \mathbf{i}| = 1$,
- $\mathbf{r} \cdot \mathbf{i} = \mathbf{r} \cdot \mathbf{j} = \mathbf{r} \cdot \mathbf{k}$,
- $\mathbf{r} \wedge \mathbf{i} = \mathbf{i}$.

2. Find the shortest distance between the lines

$$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z}{2} \quad \text{and} \quad x = 2, \quad \frac{y-1}{2} = z.$$

[Hint: parametrise the lines and write down the vector between two arbitrary points on the lines; then determine when this vector is normal to both lines.]

3. Let L_θ denote the line through (a, b) making an angle θ with the x-axis. Show that L_θ is a tangent of the parabola $y = x^2$ if and only if

$$\tan^2\theta - 4a\tan\theta + 4b = 0.$$

[Hint: parametrise L_θ as $x = a + \lambda\cos\theta$ and $y = b + \lambda\sin\theta$ and determine when L_θ meets the parabola precisely once.]

Show that the tangents from (a, b) to the parabola subtend an angle $\pi/4$ if and only if

$$1 + 24b + 16b^2 = 16a^2.$$

[Hint: use the formula $\tan(\theta_1 - \theta_2) = (\tan\theta_1 - \tan\theta_2)/(1 + \tan\theta_1\tan\theta_2)$.]

Sketch the curve $1 + 24y + 16y^2 = 16x^2$ and the original parabola on the same axes.

4. What transformations of the xy-plane do the following matrices represent:

$$i) \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad ii) \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$
$$iii) \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad iv) \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Which, if any, of these transformations are invertible?

5. The *cycloid* is the curve given parametrically by the equations

$$x(t) = t - \sin t, \quad \text{and} \quad y(t) = 1 - \cos t \quad \text{for } 0 \le t \le 2\pi.$$

(a) Sketch the cycloid.

(b) Find the arc-length of the cycloid.

(c) Find the area bounded by the cycloid and the x-axis.

(d) Find the area of the surface of revolution generated by rotating the cycloid around the x-axis.

(e) Find the volume enclosed by the surface of revolution generated by rotating the cycloid around the x-axis.