A landscape sketch of Quantum Complexity

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Cryptography and Quantum Computing
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A view of
Randomised Computation

Distributions are data,
and transform linearly

A view of

Randomised Computation

Distributions are data, and transform linearly

$x \in \{0,1\}^k$

$p(x_1, \ldots, x_k)$

$y \in \{0,1\}^r$

$p'(y_1, \ldots, y_r)$

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Distributions are data, and transform linearly

- Computations on random bits
  = linear (stochastic) transformations
  of probability distributions

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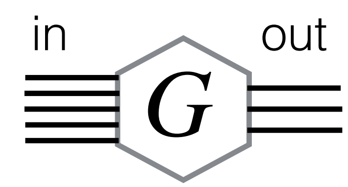

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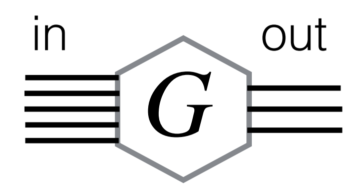
$x \in \{0,1\}^k$  $y \in \{0,1\}^r$

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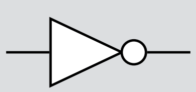
NOT p  p' $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_0 \end{bmatrix}$

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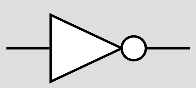
NOT  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

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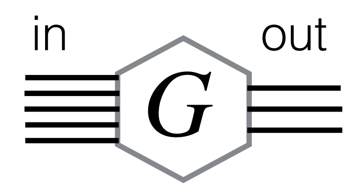
$x \in \{0,1\}^k$  $y \in \{0,1\}^r$

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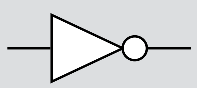
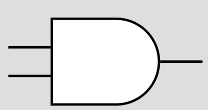
NOT  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

AND  $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{00} \\ p_{01} \\ p_{10} \\ p_{11} \end{bmatrix} = \begin{bmatrix} p_{00}+p_{01}+p_{10} \\ p_{11} \end{bmatrix}$

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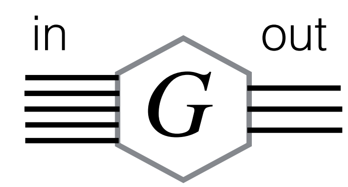
$x \in \{0,1\}^k$  $y \in \{0,1\}^r$

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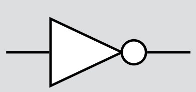
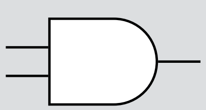
NOT  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

AND  $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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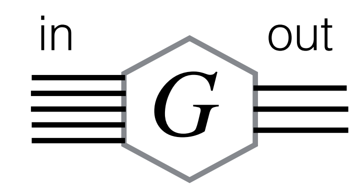
$x \in \{0,1\}^k$ 

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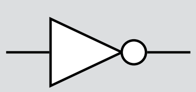
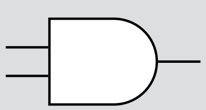
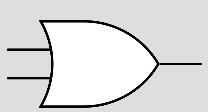
NOT		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
AND		$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
OR		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

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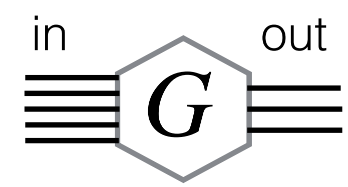
$x \in \{0,1\}^k$  $y \in \{0,1\}^r$

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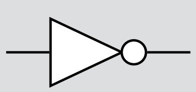
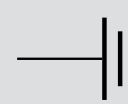
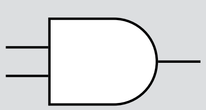
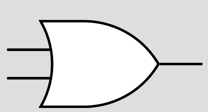
NOT		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$		ERASE		$\begin{bmatrix}1 & 1\end{bmatrix}$
AND		$\begin{bmatrix}1 & 1 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$				
OR		$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 1 & 1\end{bmatrix}$				

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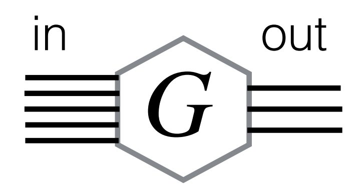
$x \in \{0,1\}^k$  $y \in \{0,1\}^r$

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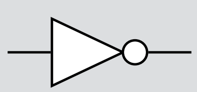
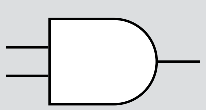
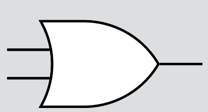
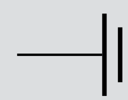
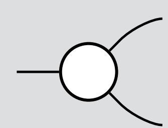
NOT  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

AND  $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

OR  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

ERASE  $\begin{bmatrix} 1 & 1 \end{bmatrix}$

FANOUT  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

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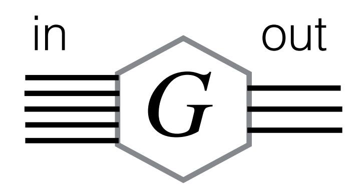
$x \in \{0,1\}^k$  $y \in \{0,1\}^r$

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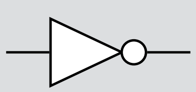
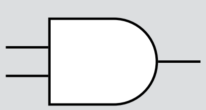
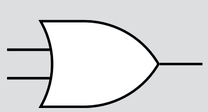
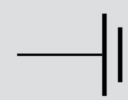
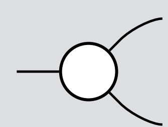
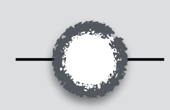
NOT		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
AND		$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
OR		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$
ERASE		$\begin{bmatrix} 1 & 1 \end{bmatrix}$
FANOUT		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$
MIX		$\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$

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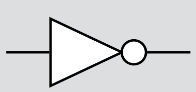
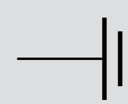
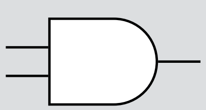
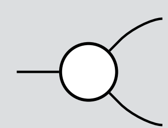
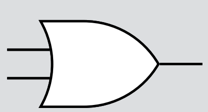
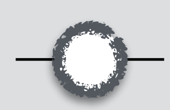
NOT		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	ERASE		$\begin{bmatrix} 1 & 1 \end{bmatrix}$
AND		$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	FANOUT		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$
OR		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$	MIX		$\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$

- Polynomial-size randomised circuits
  = efficiently describable (stochastic) tensor networks

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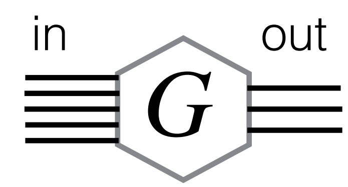
$x \in \{0,1\}^k$  $y \in \{0,1\}^r$

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Exponentially long real-valued vectors?

Exponentially small amplitudes?

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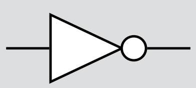
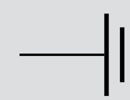
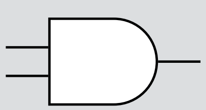
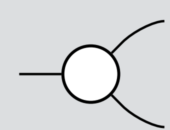
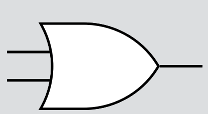
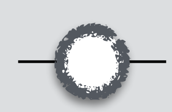
NOT		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	ERASE		$\begin{bmatrix} 1 & 1 \end{bmatrix}$
AND		$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	FANOUT		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$
OR		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$	MIX		$\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$

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Not a problem!
(these are *descriptions* of algorithms, not *products* of them)

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Distributions are data, and transform linearly

- Multi-qubit states = ℓ_2-unit 'distributions'
  over bit-strings

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Distributions are data, and transform linearly

- Multi-qubit states = ℓ_2-unit ‘distributions’ over bit-strings

$\longrightarrow$ $\uparrow$

$|0\rangle$ $|1\rangle$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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$\rightarrow$	$\uparrow$	$\nearrow$	$\searrow$			
$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert +\rangle$	$\lvert -\rangle$	$\psi(x)$	$\psi(x_1 x_2)$	*etc.*
$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$	$\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$	$\begin{bmatrix}\psi_0\\\psi_1\end{bmatrix}$	$\begin{bmatrix}\psi_{00}\\\psi_{01}\\\psi_{10}\\\psi_{11}\end{bmatrix}$	

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$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$	$\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$	$\begin{bmatrix}\psi_0\\\psi_1\end{bmatrix}$	$\begin{bmatrix}\psi_{00}\\\psi_{01}\\\psi_{10}\\\psi_{11}\end{bmatrix}$	

$$\mathbf{Pr}[x_n{=}0] = \sum_{y\in\{0,1\}^{n-1}} \left|\psi_{y;0}\right|^2$$

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Distributions are data, and transform linearly

- Multi-qubit states = ℓ_2-unit ‘distributions’ over bit-strings
- Valid operations = linear (ℓ_2-norm preserving) transformations

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A view of
Quantum Computation

Distributions are data, and transform linearly

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"Exponentially long vectors?"… no problem

Commonalities

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- Algorithms represented by tensor networks (“*circuits*”)
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- Minute **individual** coefficients are not significant
  (*i.e.* unstructured search appears to require exponential time)

Example:
quantum *lower* bound for search

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Proof idea: Bound the effect of cumulative "oracle" queries on distributions, in a protocol with a generic input state

$\implies$ quantum "parallelism" cannot directly simulate nondeterministic "parallelism"

One proof technique:

The polynomial method

An oracle $A : \{0,1\}^n \to \{0,1\}$ is represented by an (exponentially long) boolean string $A \in \{0,1\}^{2^n}$

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arXiv:quant-ph/9802049

Lemma 4.1 *Let $\mathcal{N}$ be a quantum network that makes T queries to a black-box X. Then there exist complex-valued N-variate multilinear polynomials $p_0, \ldots, p_{2^m-1}$, each of degree at most T, such that the final state of the network is the superposition*

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— similar results hold for randomised algorithms as well

Overview of lower bound techniques

in quantum *query* complexity

(adapted from `arXiv:1209.2713`)

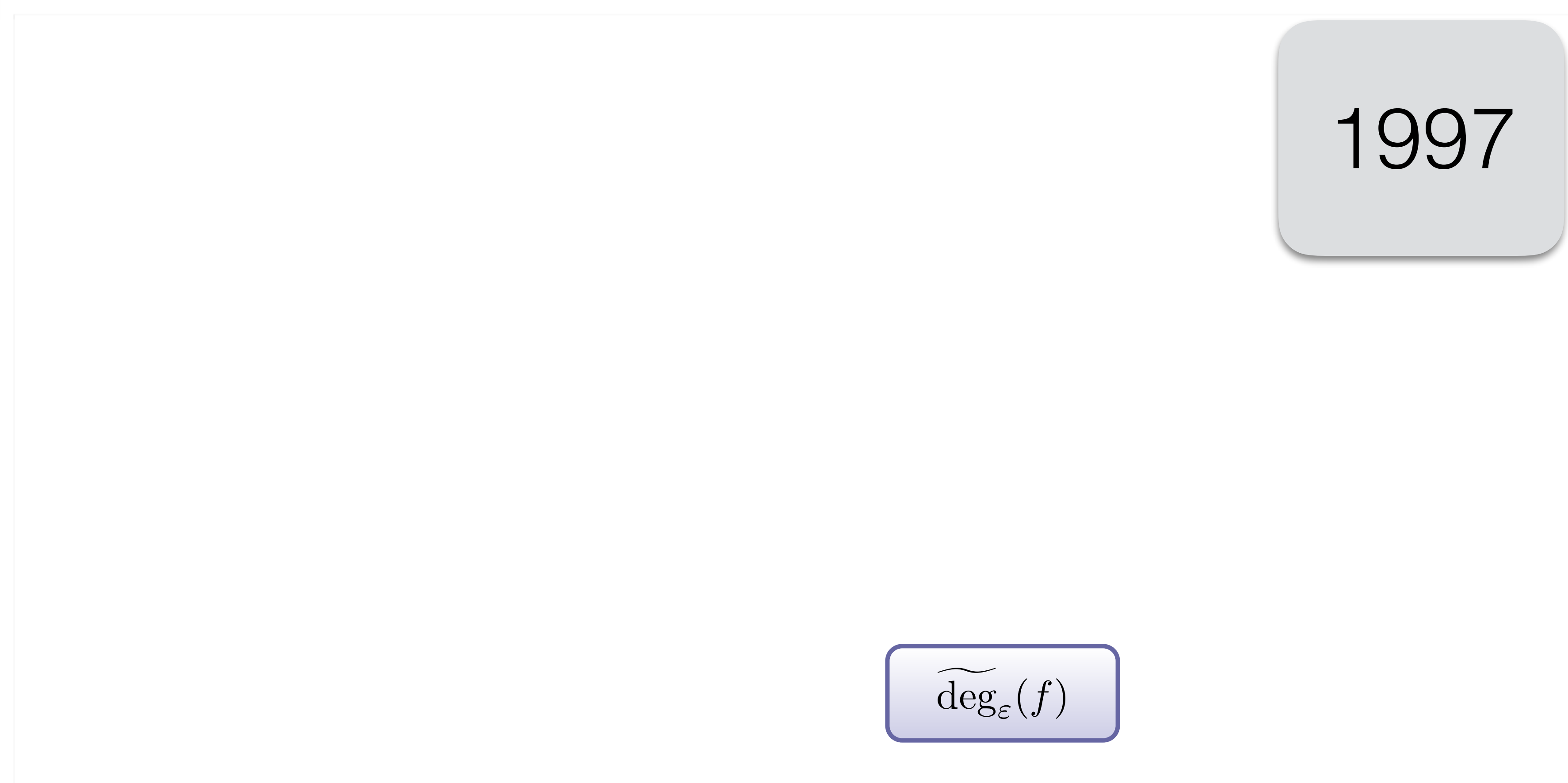

Figure 1: Relations between the different methods to prove lower bounds for quantum query complexity. An arrow from method A to method B implies that any lower bound that can be proved with A can also be proved with B (i.e., B is stronger than A).

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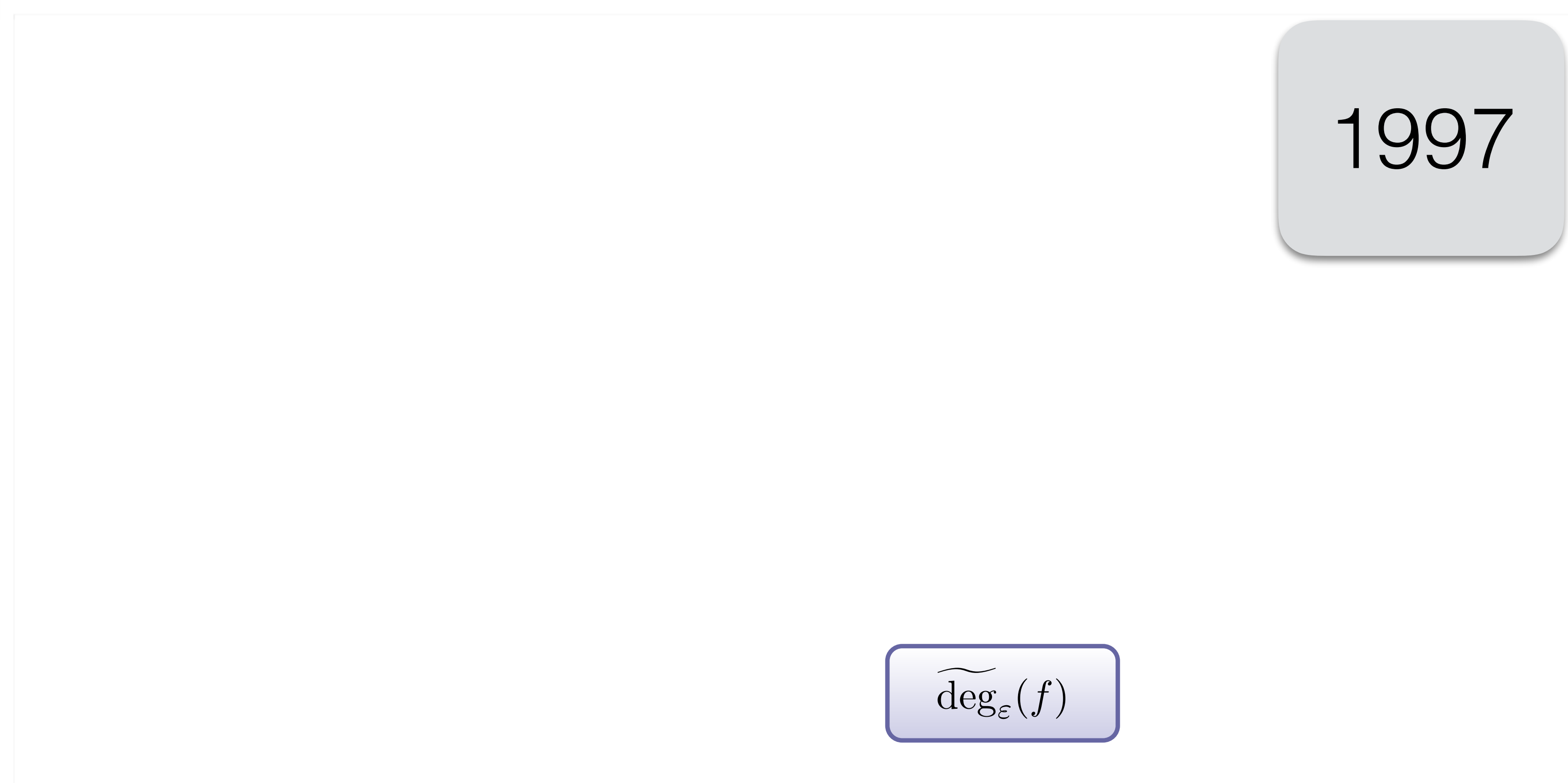
“Adversary” methods — lower-bound the query complexity, by measuring amount of work needed to distinguish inputs with different outputs



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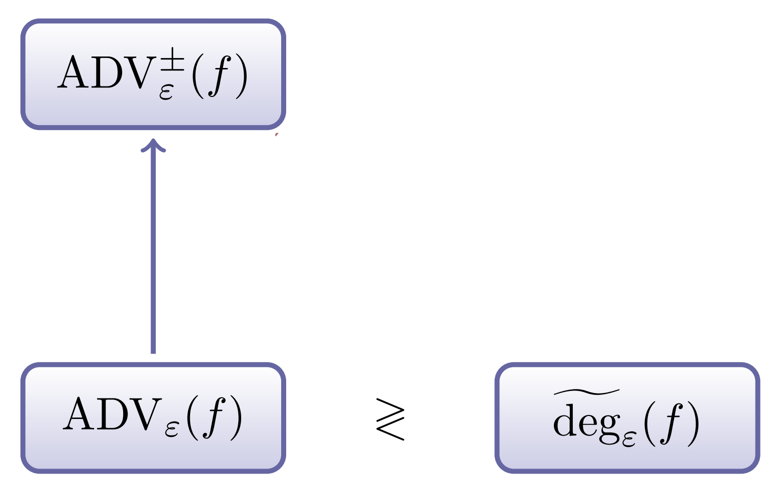
(arrows: constructive reductions)



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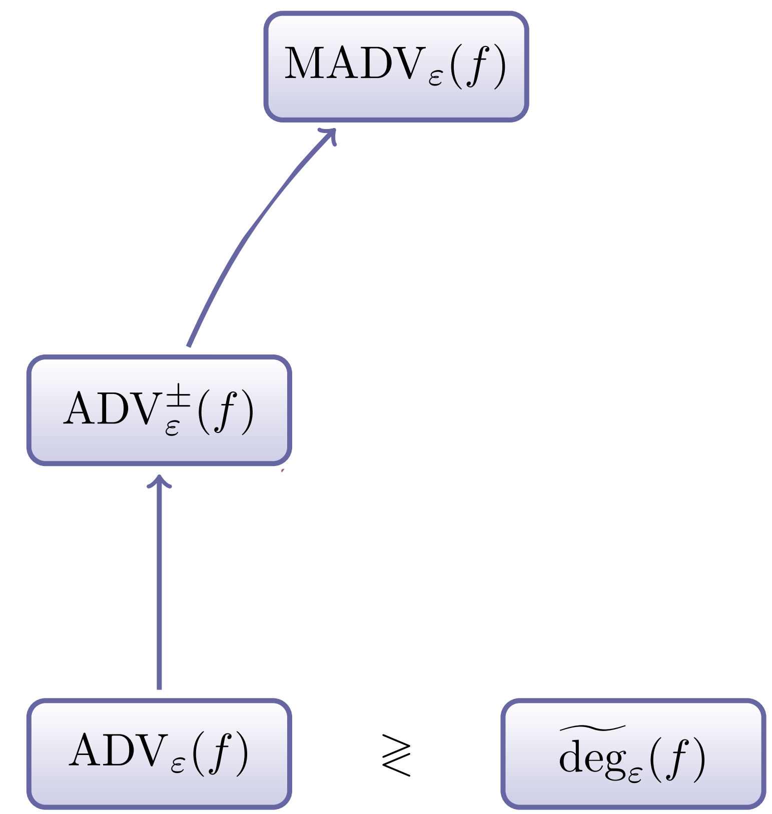

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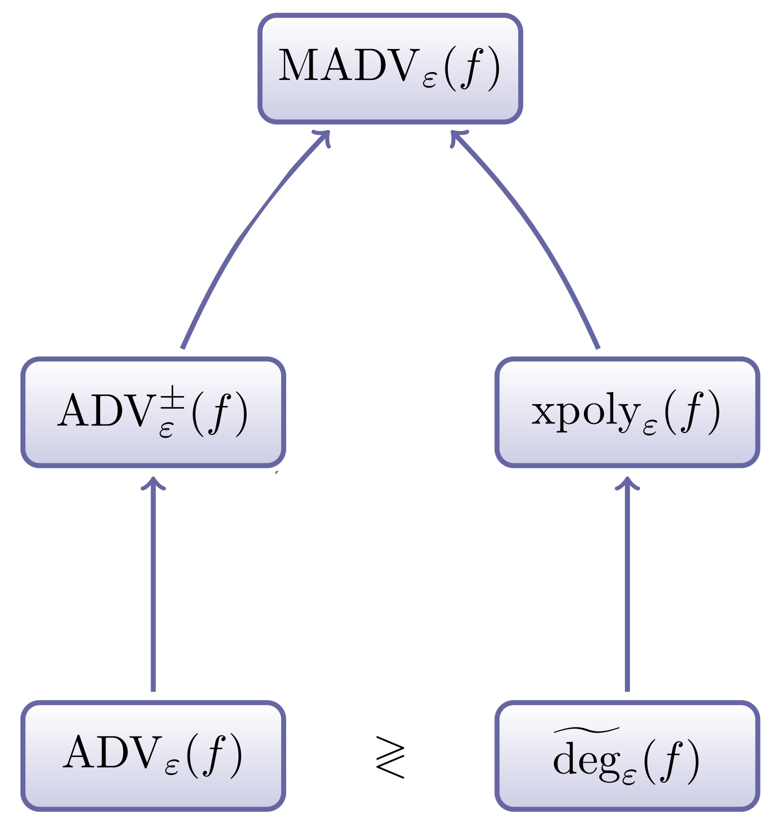

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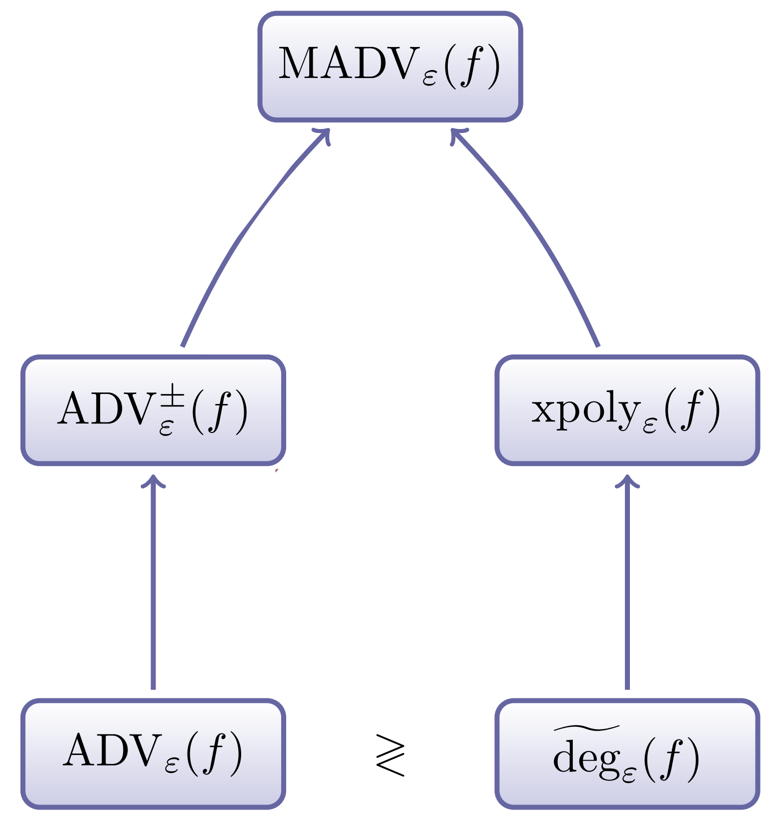
Both the "negative" and "multiplicative" adversary methods characterise quantum query complexity [`arXiv:0904.2759`]



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eg. the “eigenspace trick” (as one may call it)

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$\rightarrow$ Order finding

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[arXiv:1511.02306]

“Fourier” decomposition

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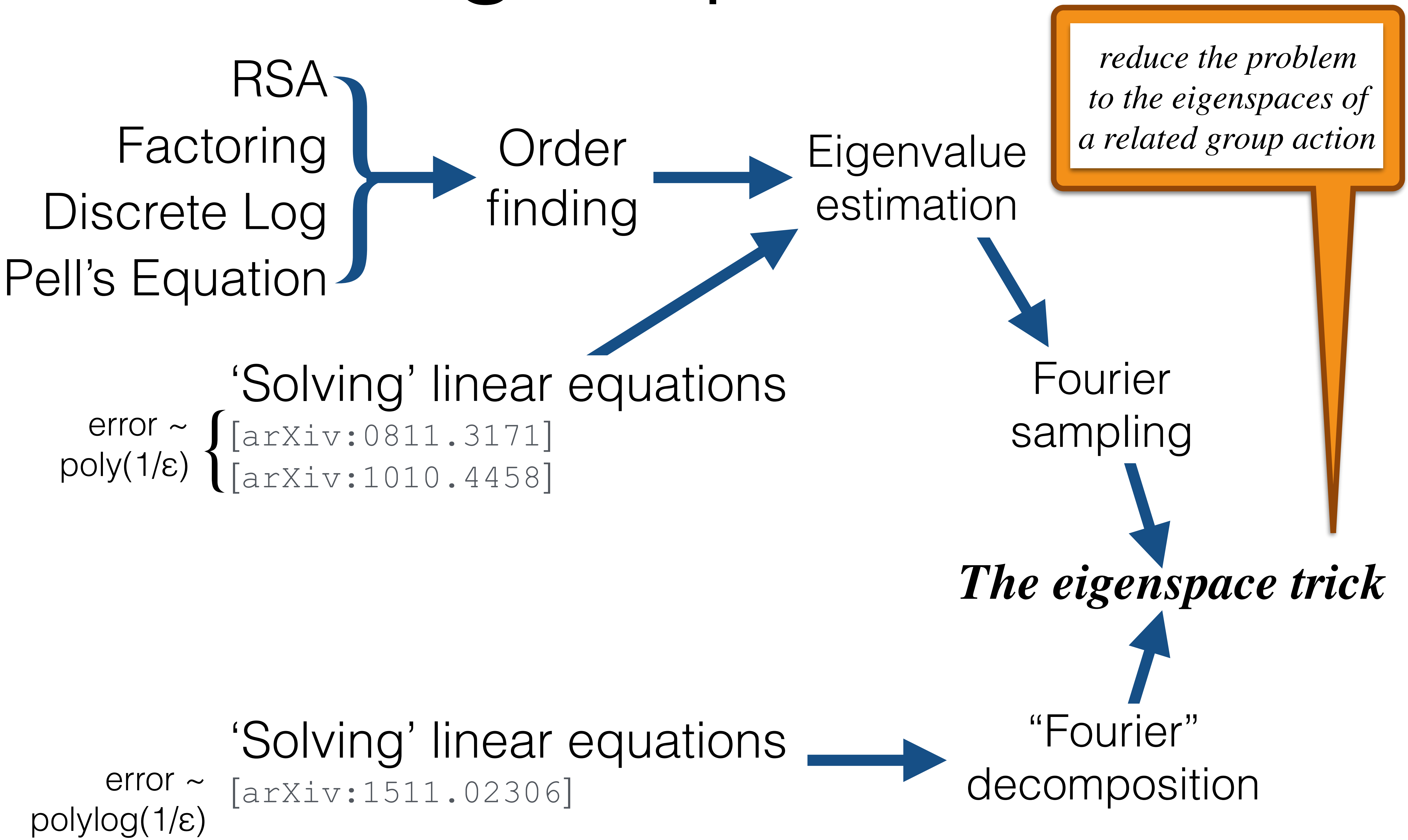

A quick description of

The "eigenspace trick" (#1)

reduce the problem to the eigenspaces of a related group action

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(4. Perform further operations conditioned on eigenvalue estimates)

(e.g. obtain a rational estimate, and find the order of A)

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2. Use the control to (coherently) perform operations from the family of commuting operators

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(4. Condition on success of erasure)

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Caveat scriptor

If problem **X** has more convenient structure than problem **Y**, useful group actions for **X** may be easier to access

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Multivariate equations: based on the difficulty of solving systems of polynomial equations in many variables

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Post-quantum question: for a given γ, are "useful" group actions hard to access for quantum computers?

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Post-quantum question: Does restricting to efficiently decodable linear codes make “useful” group actions accessible to a quantum attacker?

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Post-quantum question: Does the privately held similarity transform suffice, to hide the privately held system of equations?

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- What other strategies (beyond the eigenspace trick) may form the basis of useful quantum algorithms?