An ~~group~~ algebra of CZ transformations

We have seen that if Ω defined on the sphere is continuous with a Dini-type modulus of continuity and has mean zero, then the operator

$$\text{pv}\ \frac{\Omega(x')}{|x|^d} = \mathbb{H}_\Omega$$

has the usual nice properties. We have also computed explicitly $m = \widehat{\mathbb{H}_\Omega}$

It is a natural question trying to describe the multiplicators m arising in this way, or at least some condition on m ensuring that it is of this type. We will see some of this in the Littlewood-Paley theory. By now we state a particular case

Theorem. We consider $\Omega \in C^\infty(S^{d-1})$ with mean zero. Then the $m = \widehat{\mathbb{H}_\Omega}$ are exactly the C^∞ functions in $\mathbb{R}^d\setminus\{0\}$, homogeneous of degree 0, with mean zero

If $m \in C^\infty(\mathbb{R}^d\setminus\{0\})$ is homogeneous of degree 0, and $c = \int_{S^{d-1}} m\, d\sigma$ is its mean, then

$$m = \widehat{c\delta_0 + \mathbb{H}_\Omega}$$

for some Ω. ~~theorems means that the class of operators~~

Proof of the theorem. If $\Omega \in C^\infty(S^{d-1})$ has mean zero we already know that $m = \widehat{H_\Omega}$ is given by the formula

$$m(\xi) = \int_{S^{d-1}} \Omega(u) \left[\log \frac{1}{|u \cdot \xi'|} - i \frac{\pi}{2} \operatorname{sign}(u \cdot \xi') \right] d\sigma(u)$$

We also know that $m \in C^\infty(\mathbb{R}^d \setminus \{0\})$ by the general result on homogeneous distributions (or else can be checked directly). Using this expression we see that m has mean zero.

Conversely, assume that m is $C^\infty(\mathbb{R}^d \setminus \{0\})$, homogeneous of degree zero, with mean zero. Obviously $m \in L^1_{loc}(\mathbb{R}^d)$ and so defines a distribution, which is tempered. We consider $\hat{m}$, another tempered distribution. Then

$$(D^\alpha m)^\wedge(\xi) = (2\pi i \xi)^\alpha \, \hat{m}(\xi)$$

If $|\alpha| = d$, $D^\alpha m$ is an homogeneous function of degree $-d$ $C^\infty(\mathbb{R}^d \setminus \{0\})$ and with mean zero; for those we saw that as distributions

$$(D^\alpha m) = \text{p.v. } D^\alpha m$$

We conclude that

$$(2\pi i \xi)^\alpha \, \hat{m}(\xi) = \widehat{\text{p.v. } D^\alpha m}$$

By what we already know, $\widehat{\text{p.v. } D^\alpha m}$ is $C^\infty(\mathbb{R}^d \setminus \{0\})$ so we conclude that $\hat{m} \in C^\infty(\mathbb{R}^d \setminus \{0\})$. Since it is homogeneous of degree $-d$, ~~we can write it in the form~~

if Ω denotes its restriction to S^{d-1}, ~~mt~~ $\hat{m}(\xi)$ equals $\Omega(\xi')/|\xi|^d$ on $\mathbb{R}^d \setminus 0$.

Next we prove that Ω has mean zero. For this we pick $\varphi \in S(\mathbb{R}^d)$ radial with support in $1 \le |x| \le 2$, $\varphi > 0$ in $1 < |x| < 2$; then

$$\langle \hat{m}, \varphi \rangle = \int \frac{\Omega(\xi')}{|\xi|^d} \varphi(\xi)\, d\xi = c \int_{S^{d-1}} \Omega(\xi')\, d\sigma(\xi')$$

with $c > 0$. But

$$\langle \hat{m}, \varphi \rangle = \langle m, \hat{\varphi} \rangle = \int_{\mathbb{R}^d} m(x)\, \hat{\varphi}(x)\, dx$$

is zero because $\hat{\varphi}$ is radial and m has mean zero.

This allows us to consider H_Ω. To see that $\hat{m} = H_\Omega$ we argue as follows: the difference $\hat{m} - H_\Omega$ is a tempered distribution supported in zero, hence is a linear combination μ of derivatives of δ. But m and $\widehat{H_\Omega}$ are bounded, so $\hat{\mu}$ is a bounded polynomial, a constant, $\mu = \lambda \delta_0$, and

$$m - \widehat{H_\Omega} = \lambda$$

Since both m, $\widehat{H_\Omega}$ have mean zero, it follows that $\lambda = 0$ ~~and~~, $\hat{m} = H_\Omega$ and so $m = H_{\tilde{\Omega}}$ (where $\tilde{\Omega}(x) = +\Omega(-x)$). //

If $m \in C^\infty(\mathbb{R}^d \setminus 0)$ is homogeneous of degree 0 but has nonzero mean c, then $m - c = \widehat{H_\Omega}$ hence

$$m = \widehat{c\delta + H_\Omega}$$

This means that the class of operators

given by

$$\widehat{Tf}(\xi) = m(\xi)\,\hat{f}(\xi) \quad , f \in S(\mathbb{R}^d)$$

with $m \in C^\infty(\mathbb{R}^d \setminus \{0\})$ homogeneous of degree 0 is exactly equal to the class of operators given by

$$Tf(x) = c\, f(x) + \text{p.v.} \int f(x-y) \frac{\Omega(y')}{|y|^n}\, dy$$

This is an <u>algebra</u> $\mathcal{A}$ of operators, and T is invertible iff $m \neq 0$ at all points.

A particularly important example are the <u>elliptic</u> homogeneous partial differential operators with constant coefficients

$$P(D) f = \sum_{|\alpha| = m} a_\alpha D^\alpha f$$

whose symbol is given by

$$\widehat{P(D) f}(\xi) = P(2\pi i \xi)\,\hat{f}(\xi)$$

Let $\Lambda = \sqrt{-\Delta}$, that is, $\widehat{\Lambda f}(\xi) = 2\pi|\xi|\,\hat{f}(\xi)$.
Then we can write $P(D) f = T(\Lambda^m f)$ with

$$\widehat{Tf}(\xi) = \frac{P(2\pi i \xi)}{(2\pi|\xi|)^m}\,\hat{f}(\xi) = i^m \frac{P(\xi)}{|\xi|^m}\,\hat{f}(\xi)$$

T is invertible in the algebra if and only $P(\xi)$ does not vanish in S^{d-1}, that is, P is elliptic (and then m is even $m = 2k$). Then the problem

$$P(D) u = f$$

is equivalent to $(-\Delta)^k u = T^{-1} f$

A consequence of this is that if P is ellyptic then

$$1<p<\infty \qquad \|D^\alpha \varphi\|_p \leq C_p \|P(D)\varphi\|_p, \quad \varphi \in C_c^\infty(\mathbb{R}^d)$$

for all multiindexes α, $|\alpha| \leq m$. The operator that goes from $P(D)\varphi$ to $D^\alpha\varphi$ has symbol

$$m(\xi) = \frac{(2\pi i \xi)^\alpha}{P(2\pi i \xi)}$$

in this algebra.

The Beurling transform

The Beurling transform is a CZO of convolution type in $\mathbb{C} = \mathbb{R}^2$ that plays an important role in complex analysis and quasiconformal theory. Let us give a glimpse to it.

We use the notations $z = x + iy$

$$\frac{\partial}{\partial \bar{z}} = \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial z}$$

Functions f with $\bar{\partial} f = 0$ (Cauchy-Riemann eqs) are <u>holomorphic</u> or conformal, locally sums of $\sum a_n z^n$

Functions with $\partial f = 0$ are the conjugates of holomorphic, called antiholomorphic or anticonformal.

It is immediate to check that

$$4 \partial \bar{\partial} = \Delta$$

Consider the fundamental solution $\frac{1}{2\pi} \log |z| = \frac{1}{4\pi} \log z\bar{z}$ of Δ. That is

$$\frac{1}{\pi} \partial \bar{\partial} \log z\bar{z} = \delta_0$$

Since $\partial \log |z|^2 = \partial \log z\bar{z} = \frac{1}{z}$ we have

$$\bar{\partial}\left(\frac{1}{\pi}\frac{1}{z}\right) = \delta_0$$

which means that $\frac{1}{\pi z}$ is the fundamental solution of $\bar{\partial}$

(Cauchy transform) $$\varphi(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar\partial\varphi(w)}{z-w}\, dA(w), \quad \varphi \in C_c^\infty(\mathbb{C})$$

In fact the companion formula of this is Cauchy reproducing formula

$$\varphi(z) = \frac{1}{2\pi i}\int_{\partial U} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi}\int_U \frac{\bar\partial\varphi(w)}{w-z}\, dA(w)$$

We compute now some Fourier transforms. We use $\xi =$ for the dual variable of $z = x+iy$. First note that

$$\widehat{\bar\partial f}(\xi) = \xi \hat f(\xi), \quad \widehat{\partial f}(\xi) = \bar\xi \hat f(\xi)$$

and hence

$$\widehat{\frac{1}{\pi}\frac{1}{z}} = \frac{1}{\xi}$$

The operator that maps $\bar\partial f$ to ∂f

$$\widehat{\bar\partial f} = \xi \hat f(\xi) \longrightarrow \widehat{\partial f} = \bar\xi \hat f$$

is the one with multiplicator $m = \frac{\bar\xi}{\xi} = \frac{\bar\xi^2}{|\xi|^2}$

This operator is in our algebra $\mathcal{A}$. Applying (formally) ∂ to

$$\widehat{\frac{1}{\pi}\frac{1}{z}} = \frac{1}{\xi}$$

we see that on the convolution side it corresponds to

$$\text{p.v.} -\frac{1}{\pi}\frac{1}{z^2}$$

$$\widehat{-\text{p.v.}\frac{1}{\pi}\frac{1}{z^2}} = \frac{\bar{\xi}}{\xi}$$

This is confirmed applying ∂ to the formula above

$$\frac{\partial \varphi}{\partial z}(z) = -\frac{1}{\pi}\text{p.v.}\int_{\mathbb{C}} \frac{\bar{\partial}\varphi(w)}{(z-w)^2}\, dA(w)$$

The operator

$$Bf(z) = -\frac{1}{\pi}\ \text{p.v.} \int \frac{f(w)}{(z-w)^2}\, dA(w)$$

is called the <u>Beurling transform</u>, and is a CZO of convolution type, homogeneous.

Since $|m(\xi)| = \left|\frac{\bar{\xi}^2}{|\xi|^2}\right| = 1$, $\|Bf\|_2 = \|f\|_2$.

In particular

$$\int |\bar{\partial} f|^2\, dA = \int |\partial f|^2\, dA$$

(something that it is trivial by integration by parts)
We know thus that

$$\|Bf\|_p \le C_p \|f\|_p \qquad f \in L^p(\mathbb{C}),\ 1<p<\infty$$

$$\|\partial\varphi\|_p \le C_p \|\bar\partial\varphi\|_p \qquad \varphi \in C_c^\infty(\mathbb{C})$$

If we express the above in terms of $\varphi = u + iv$

$$2\bar\partial\varphi = (u_x + iv_x + i(u_y + iv_y)) = (u_x - v_y) + i(u_y + v_x)$$

$$2\partial\varphi = (u_x + iv_x - i(u_y + iv_y)) = (u_x + v_y) + i(v_x - u_y)$$

we find the statement that $u_x + v_y$, $v_x - u_y$ are controlled by $u_x - v_y$, $u_y + v_x$ while Korn's inequality says that

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \text{ is controlled by } \begin{pmatrix} u_x & \frac{u_y + v_x}{2} \\ \frac{u_y + v_x}{2} & v_y \end{pmatrix}$$

The Beurling transform plays an important role in the theory of quasiconformal maps

A fast survey about quasiconformal (QC) maps (Ahlfors notes)

Conformal transformations map infinitesimal circles to infinitesimal circles and are sense preserving: this is because $df(z)$ is simply multiplication by $f'(z)$

QC maps are essentially maps that have bounded distorsion and sense preserving. To measure distorsion the complex notation is very convenient

$$df = \frac{\partial f}{\partial \bar z} d\bar z + \frac{\partial f}{\partial z} dz,$$

If $f = u + iv$,

$$\partial f = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y)$$

$$\bar\partial f = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y)$$

Then

$$|\partial f|^2 - |\bar\partial f|^2 = u_x v_y - u_y v_x = J(f) \quad \text{jacobian}$$

$J > 0$ for sense preserving, $J < 0$ for reversing. We shall consider $J > 0$, $|\bar\partial f| < |\partial f|$. If $w = f(z)$ we have

$$|\partial f| - |\bar\partial f| \leq |dw| \leq (|\partial f| + |\bar\partial f|)|dz|$$

So the ratio of the major linear distorsion to the minor is

$$D_f = \frac{|\partial f| + |\bar\partial f|}{|\partial f| - |\bar\partial f|} \geq 1$$

This is called the dilatation at z. One uses too

$$d_f(z) = \frac{|\bar\partial f|}{|\partial f|} < 1$$

So that

$$D_f = \frac{1+d_f}{1-d_f} \qquad d_f = \frac{D_f - 1}{D_f + 1}$$

Note that f is conformal at z iff $D_f = 1$, $d_f = 0$.

The maximum is attained when $\dfrac{\bar\partial f \, d\bar z}{\partial f \, dz}$ is positive, the minimum when is negative. For this reason one introduces the complex dilatation

$$\mu_f = \frac{\bar\partial f}{\partial f}$$

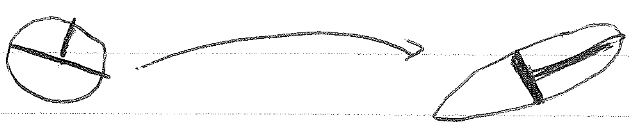
The maximum dilation distortion is in the direction of

$$\arg dz = \frac{1}{2} \arg \mu_f = \alpha$$

and the minimum in the direction $\alpha \pm \frac{\pi}{2}$. In the w plane, $w = f(z)$, the direction of the major axis is

$$\arg dw = \arg \partial f \, dz \left(1 + \frac{\bar\partial f \, d\bar z}{\partial f \, dz}\right) =$$

$$= \arg \partial f \, dz = \frac{1}{2} \arg \nu_f \quad , \nu_f = \left(\frac{\partial f}{|\partial f|}\right)^2 \mu_f =$$



<u>Definition</u>. A ~~map~~ homeomorphism $f: \Omega \to \Omega'$ is called K-quasiconformal if it is orientation preserving $f \in W^{1,2}_{loc}(\Omega)$ and if

$$|\mu_f(z)| \le k < 1 \quad \text{a.e. } z \in \Omega.$$

We say that f is quasiconformal from Ω to Ω' (QC)

One of the most important results in QC theory is the so called the <u>measurable Riemann mapping theorem</u> on the existence of QC maps with <u>prescribed</u> complex dilatation μ.

One should think that giving the data μ means geometrically to give a field of infinitesimal elipses, one at each point. The theorem says then than changing coordinates one may view all elipses as circles. The precise statement is

<u>Theorem</u> Given $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty = k < 1$ there exists a unique QC map f^μ normalized (meaning that leaves $0, 1, \infty$ fixed) such that

$$\bar\partial f(z) = \mu(z)\, \partial f(z)$$

The above equation is called the <u>Beltrami equation</u>.

We will sketch the proof of this theorem in one particular case, that shows the role of the Beurling transform. Before that, let us ~~make~~ recall some facts about the Beurling transform: we know that

$$\|B\varphi\|_p \leq C_p \|\varphi\|_p \qquad 1<p<\infty, \ \varphi \in C_c^\infty(\mathbb{C})$$

so C_p is the norm of B as a bded operator in $L^p(\mathbb{C})$. Since it is self-adjoint, one has

$$C_p = C_q \quad \text{if} \quad \frac{1}{p}+\frac{1}{q}=1.$$

Also $C_2 = 1$ (isometry of $L^2(\mathbb{C})$), and evidently $C_p \to 1$ as $p \to 2$.

The behaviour of C_p as $p \to 2$ turns out to be a very important point dealing with regularity properties of QC maps. It is an open problem to show that

$$C_p = \begin{cases} p-1 & p \geq 2 \\ \dfrac{1}{p-1} & p \leq 2 \end{cases}$$

<u>Particular case of uniformization theorem</u>: if $\mu \in L^\infty(\mathbb{C})$ has compact support and $\|\mu\|_\infty \leq k < 1$ there is a unique f such that $f(0)=0$, $\partial f(z) - 1 \in L^p$ and

$$\bar\partial f(z) = \mu(z)\, \partial f(z)$$

<u>Proof</u>. We start choosing $2<p$ such that $k\,C_p<1$

We consider for convenience a modification (by a constant) of the Cauchy transform

$$Ph(z)=\frac{1}{\pi}\int_{\mathbb{C}}\frac{h(w)}{z-w}\,dA(w)+\frac{1}{\pi}\int\frac{h(w)}{w}\,dA(w)=$$

$$=\frac{z}{\pi}\int\frac{h(w)}{w(z-w)}\,dA(w)\quad;\quad\bar\partial Ph=h$$

One may check that Ph makes sense for $h\in L^p$, $p>2$, and Ph satisfies a Lipschitz condition with exponent $1-2/p$

We know
$$\bar\partial Ph=h$$
$$\partial Ph=Bh$$

<u>Unicity</u>. Assume $\bar\partial f=\mu\,\partial f$, then $\bar\partial f\in L^p(\mathbb{C})$ can consider $P(\bar\partial f)$. Then $F=f-P(\bar\partial f)$ is holomorphic, entire and

$$F'=\partial F=\partial f-\partial P\bar\partial f=\partial f-B(\bar\partial f)$$

and so $F'-1\in L^p(\mathbb{C})$, which implies $F'=1$, $F=z+a$

The normalization $f(0)=0$ implies $F(0)=0$ so $a=0$ and

$$f=P(f_{\bar z})+z$$

and so

$$\partial f=B(\bar\partial f)+1=B\mu\,\partial f+1$$

If g is another solution, $\partial g=B\mu\,\partial g+1$ and

$$\partial f-\partial g=B[\mu(\partial f-\partial g)]$$

Then by the L^p-estimate of B

$$\|\partial f - \partial g\|_p \leq C_p \|\mu\|_\infty \|\partial f - \partial g\|_p$$

Since $\|\mu\|_\infty C_p < 1$ we must have $\partial f = \partial g$ a.e and then also $\bar\partial f = \bar\partial g$, $f-g$ is constant and by the normalization $f = g$

<u>Existence</u> We consider the equation

$$h = B(\mu h) + B\mu \qquad (I - B\mu)h = B\mu$$

Using $\|\mu B\| \leq \|\mu\|_\infty C_p < 1$ we see that $(I - B\mu)$ is invertible as an operator in L^p with inverse

$$(I - B\mu)^{-1} = \sum_{n=0}^{\infty} (B\mu)^n = I + B\mu + B\mu B\mu + \cdots$$

Hence a solution of $h = B(\mu h) + B\mu$ is

$$h = (I - B\mu)^{-1} B\mu = B\mu + B\mu B\mu + \cdots \in L^p(\mathbb{C})$$

We prove now that $f = P[\mu(h+1)] + z$ is a solution of the Beltrami equation

$\mu(h+1) \in L^p$ (because μ has compact support) $\Rightarrow$
$\Rightarrow P[\mu(h+1)]$ continuous. Then

$$\bar\partial f = \mu(h+1)$$
$$\partial f = B[\mu h + \mu] + 1 = h + 1$$

and $\partial f - 1 = h \in L^p$ //

Singular integrals with variable kernel

Let $P(x,D)$ homogeneous with variable coeffs

$$P(x,D) = \sum_{|\alpha|=m} b_\alpha(x) D^\alpha$$

$$D^\alpha f(x) = \int (2\pi i \xi)^\alpha \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi$$

$$\rightarrow P(x,D) f(x) = \int P(x, 2\pi i \xi) \hat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi$$

If $\Lambda = \sqrt{-\Delta}$ (with symbol $2\pi|\xi|$), again

$$P(x,D) f = T(\Lambda^m f)$$

with

$$Tf(x) = \int \sigma(x,\xi) \hat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi$$

variable symbol

where

$$\sigma(x,\xi) = \frac{P(x, i\xi)}{|\xi|^m}$$

σ is homogeneous in ξ of degree zero.

Now we replace $\hat{f}(\xi)$ by its expression

$$Tf(x) = \int \sigma(x,\xi) \int f(y)\, e^{-2\pi i y\xi}\, dy\, e^{2\pi i x\cdot\xi}\, d\xi =$$

$$= \int f(y) \underbrace{\int \sigma(x,\xi)\, e^{2\pi i (x-y)\cdot\xi}\, d\xi}_{K(x,\, x-y)}\, dy$$

Since $\sigma(x,\xi)$ is C^∞ in ξ and homogeneous of degree zero we know that there exists $a(x)$ and $\Omega_x \in C^\infty(\mathbb{R}^d \setminus \{0\})$, $\int_{S^{d-1}} \Omega_x \, d\sigma = 0$ such that

$$K(x,z) = a(x)\,\delta(z) + \text{p.v.}\ \frac{\Omega(x,z')}{|z|^d}$$

This is the motivation to study singular integrals with variable kernel

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{\Omega(x,y')}{|y|^d} f(x-y)\,dy$$

One can apply the general CZ theory but it is better to use here another method, the method of rotations.

The method of rotations

Let T a one-dimensional operator bounded in $L^p(\mathbb{R})$ and $u \in S^{d-1}$ a direction in $\mathbb{R}^d$. We may define an operator T_u bounded in $L^p(\mathbb{R}^d)$ (with the same norm) as follows

$$L_u = \mathbb{R} u \qquad L_u^\perp$$

$$x = x_1 u + \bar{x} \quad , \quad \bar{x} \perp u$$

We define $T_u f(x)$ as $T g(x_1)$ where $g(y_1) = f(y_1 u + \bar{x})$. Then by Fubini

$$\int_{\mathbb{R}^d} |T_u f(x)|^p dx = \int_{L_u^\perp} \int_{\mathbb{R}} |T g(x_1)|^p dx_1 \, d\bar{x} \leq$$

$$\leq C_p^p \int_{L_u^\perp} \int_{\mathbb{R}} |f(y_1 u + \bar{x})|^p dy_1 \, d\bar{x} = C_p^p \|f\|_p^p$$

For instance, the directional H-L maximal function

$$M_u f(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x - tu)| dt$$

or the directional Hilbert transforms

$$H_u f(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|t|>\varepsilon} f(x - tu) \frac{dt}{t}$$

The continuous Minkowski inequality gives

<u>Prop</u> If $\Omega \in L^1(S^{d-1})$ the operator

$$T_\Omega f(x) = \int_{S^{d-1}} \Omega(u)\, T_u f(x)\, d\sigma(u)$$

is bded in $L^p(\mathbb{R}^d)$ with norm $\leq C_p \|\Omega\|_1$

The question is to see which operators admit this representation

<u>Example 1</u>. For $\Omega \in L^1(S^{d-1})$

$$M_\Omega f(x) = \sup \frac{1}{|B(0,R)|} \int_{B(0,R)} |\Omega(y')|\,|f(x-y)|\,dy$$

in polar coordinates

$$M_\Omega f(x) = \sup_R \frac{1}{|B(0,1)|\, R^d} \int_{S^{d-1}} |\Omega(u)| \int_0^R |f(x-ru)|\, r^{n-1}\, dr\, d\sigma(u)$$

$$\leq \frac{1}{|B(0,1)|} \int_{S^{d-1}} |\Omega(u)|\, M_u f(x)\, d\sigma(u)$$

shows that M_Ω is bded in $L^p(\mathbb{R}^d)$.

Next we consider again $H_\Omega f(x) = \lim_\varepsilon \int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^d} f(x-y)\, dy$. We saw that if Ω is Dini continuous and $\int \Omega\, d\sigma = 0$ then H_Ω is a CZO. Now we study again it with minimal assumptions: put $\Omega \in L^1(S^{d-1})$ $\int \Omega\, d\sigma = 0$.

Example 2 We consider again $\Omega \in L^1(S^{d-1})$

$$\mathbb{H}_\Omega f(x) = \lim_{\varepsilon} \int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^d} f(x-y)\, dy$$

We saw that if $\int \Omega \, d\sigma = 0$ then

$$\widehat{\mathbb{H}}_\Omega(\xi) = \int_{S^{d-1}} \Omega(u) \left[\log \frac{1}{|u\cdot \xi'|} - i\frac{\pi}{2} \operatorname{sgn}(u\cdot\xi') \right] d\sigma(u)$$

~~This allows to improve what we said before~~

The two terms have a different character: the first is not bounded in ξ (under the only hypothesis that $\Omega \in L^1(S^{d-1})$) but is zero if Ω is odd. The second is bounded and zero if Ω is even. Noticing that any power of log is integrable we thus see, using the decomposition

$$\Omega_e = \frac{1}{2}(\Omega(u)+\Omega(-u)), \quad \Omega_o = \frac{1}{2}(\Omega(u)-\Omega(-u))$$

that

Corollary $\int \Omega \, d\sigma = 0$, $\Omega \in L^1(S^{d-1})$ with $\Omega_e \in L^q(S^{d-1})$ for some $q > 1 \Rightarrow \widehat{\mathbb{H}}_\Omega$ bounded.

In particular, for Ω odd, $\widehat{\mathbb{H}}_\Omega$ is bounded and so T_Ω is bded in $L^2(\mathbb{R}^d)$.

Theorem If Ω is odd, $\mathbb{H}_\Omega$ is bded in $L^p(\mathbb{R}^d)$, $1<p<\infty$

Proof
$$\mathbb{H}_\Omega f(x) = \lim_{\varepsilon\to 0} \int_{S^{d-1}} \Omega(u) \int_\varepsilon^\infty f(x-ru)\frac{dr}{r}\, d\sigma(u)$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2} \int_{S^{d-1}} \Omega(u) \int_{|r|>\varepsilon} f(x-ru)\frac{dr}{r}\, d\sigma(u) =$$

$$= \frac{\pi}{2} \int_{S^{d-1}} \Omega(u)\, H_u f(x)\, d\sigma(u)$$

The same can be done with the maximal operator (Ω odd)

$$H^*_\Omega f(x) = \sup_\varepsilon \left| \int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^d} f(x-y)\, dy \right|$$

$$\leq \frac{\pi}{2} \int_{S^{d-1}} |\Omega(u)|\, H^*_u f(x)\, d\sigma(u)$$

In case Ω is even, the above does not work. But using the Riesz transforms R_j that satisfy $\sum_j R_j^2 = -I$ we have

$$H_\Omega f = -\sum_j R_j^2 (Tf) = -\sum_j R_j (R_j H_\Omega) f$$

The idea is that $R_j T$ should have an odd kernel because is a composition of one even and one odd.

~~So one should see that if T_Ω has an odd kernel then $R_j T_\Omega$ is an operator of type $T_{\tilde\Omega}$ with $\tilde\Omega$~~

So we need to show that if Ω is even with $\Omega \in L^q(S^{d-1})$ for some $q>1$, then $R_j H_\Omega$ is of the form $T_{\tilde\Omega}$ with $\tilde\Omega$ odd and integrable.

We know that if Ω is $C^\infty(\mathbb{R}^d \setminus 0)$ then indeed

$R_j H_\Omega = H_{\tilde\Omega}$ with $\tilde\Omega \in C^\infty(\mathbb{R}^d \setminus 0)$, because

$$\cancel{\ldots}\, H_\Omega \longmapsto \cancel{\ldots}\, \widehat{H_\Omega} = m$$

is a one-to-one correspondence between the H_Ω with $\Omega \in C^\infty(\mathbb{R}^d \setminus \{0\})$, $\int \Omega \, d\sigma = 0$ and the m homogeneous of degree 0, $C^\infty(\mathbb{R}^d \setminus \{0\})$, with mean zero.

If $m = \widehat{H_\Omega}$ then the symbol of $R_j H_\Omega$ is up to a constant $m_j = \frac{\xi_j}{|\xi|} m$. If Ω is of class C^∞ in $\mathbb{R}^d \setminus \{0\}$ then ~~$\frac{\xi_j}{|\xi|} m$~~ too. If Ω is even then $m = \widehat{H_\Omega}$ is given by

$$m(\xi) = \int_{S^{d-1}} \Omega(u) \log \frac{1}{|\xi \cdot u|} \, d\sigma(u)$$

and so m is even too. Then $m_j = \frac{\xi_j}{|\xi|} m$ has integral zero too, and so $m_j = H_{\tilde{\Omega}}$ with $\tilde{\Omega} \in C^\infty(\mathbb{R}^d \setminus \{0\})$ with mean zero.

The question is then to show with a regularization argument that this continues to hold ~~with~~ for Ω even, $\Omega \in L^q(S^{d-1})$ $q > 1$ for some $q > 1$, $\int \Omega \, d\sigma = 0$. Hence

Thm. If Ω ~~is even~~ $\in L^q(S^{d-1})$ is even, (with some $q > 1$), then H_Ω is bded in all $L^p(\mathbb{R}^d)$.

Thm. If $\Omega \in L^1(S^{d-1})$ has mean zero and its even part is in $L^q(S^{d-1})$ for some $q > 1$, then H_Ω is bounded in $L^p(\mathbb{R}^d)$ for all p, $1 < p < \infty$.

We have seen that the method of rotations works particularly well for odd kernels. This is the case too for the homogeneous singular integrals with variable kernel

$$T_\Omega f(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{\Omega(x, y')}{|y|^d} f(x-y)\, dy$$

<u>Theorem</u>. Assume that $\Omega(x,y)$ is homogeneous of degree 0 in y, and

(a) $\Omega(x,-y) = -\Omega(x,y)$

(b) $\int_{S^{d-1}} \Omega(x, u)\, d\sigma(u) = 0$ a.e. x.

(c) $\Omega^*(u) = \sup_x |\Omega(x,u)| \in L^1(S^{d-1})$

Then T_Ω is bounded in $L^p(\mathbb{R}^d)$, $1<p<\infty$.

<u>Proof</u>. We just repeat the proof above, using polar coordinates

$$T_\Omega f(x) = \frac{\pi}{2} \int_{S^{d-1}} \Omega(x,u)\, H_u f(x)\, d\sigma(u)$$

Then

$$|T_\Omega f(x)| \le \frac{\pi}{2} \int_{S^{d-1}} \Omega^*(u)\, |H_u f(x)|\, d\sigma(u)$$

and we are done by continuous Minkowski ineq.

In this representation we can use ~~...~~ Holder

$$|T_\Omega f(x)| \le \frac{\pi}{2}\left(\int_{S^{d-1}} |\Omega(x,u)|^q \, d\sigma(u)\right)^{1/q}\left(\int_{S^{d-1}} |H_u f(x)|^{q'} \, d\sigma(u)\right)^{1/q'}$$

So if $\sup_x \left(\int_{S^{d-1}} |\Omega(x,u)|^q \, d\sigma(u)\right)^{1/q} = B_q < +\infty$ ~~...~~ (and hence for $1 < p < q$ as well) we see that ~~see that~~

$$|T_\Omega f(x)|^{q'} \le C \int_{S^{d-1}} |H_u f(x)|^{q'} \, d\sigma(u)$$

and so T_Ω is bounded in $L^{q'}$ (and so ~~...~~ in $L^{p'}$ as well), that is, if

$$\sup_x \left(\int_{S^{d-1}} |\Omega(x,u)|^q \, d\sigma(u)\right)^{1/q} = B_q < +\infty$$

then T_Ω is bounded in $L^p(\mathbb{R}^d)$ for all p, $q' \le p < \infty$.