The vector valued CZ operators

For the Littlewood-Paley Theory we need the vector-valued version of the main CZO theorem.

We review before some basics from Banach spaces.

— × — ×—

Let B be a separable Banach space. A function $F:\mathbb{R}^d \to B$ is named strongly measurable if for each b' in the dual space B^*, the function

$$x \mapsto \langle b', F(x)\rangle$$

is measurable. Then so is $\|F(x)\|_B$ and we define $L^p(B)$ by

$$\|F\|_p = \left(\int_{\mathbb{R}^d} \|F(x)\|_B^p \, dx\right)^{1/p} < +\infty \quad , 1 \le p < \infty$$

$$\|F\|_\infty = \operatorname*{esssup}_x \|F(x)\|_B$$

If $f:\mathbb{R}^d \to \mathbb{C}$ is scalar and $b \in B$, $f\cdot b$ denotes the function $(f\cdot b)(x) = f(x)\, b$, with norm

$$\|f\cdot b\|_p = \|f\|_p \|b\|_B$$

The linear span of the $f\cdot b$'s is called $L^p \otimes B$ and is dense in $L^p(B)$ if $1 \le p < \infty$.

For $F = \sum_j f_j \cdot b_j \in L^1 \otimes B$ the integral is defined

$$\int_{\mathbb{R}^d} F\,dx = \sum_j \left(\int_{\mathbb{R}^d} f_j(x)\,dx\right) b_j \in B$$

and the definition extends by continuity to $L^1(B)$. Alternatively we may think that

$$\int_{\mathbb{R}^d} F\,dx$$

is the unique element of B such that $\forall b' \in B^*$

$$\langle b', \int_{\mathbb{R}^d} F(x)\,dx \rangle = \int_{\mathbb{R}^d} \langle b', F(x)\rangle\,dx$$

If $F \in L^p(B)$, $G \in L^{p'}(B^*)$ with p, p^* conjugate exponents then

$$\langle F, G\rangle(x) = \langle F(x), G(x)\rangle$$

is integrable and

$$\|G\|_{L^{p'}(B^*)} = \sup\left\{ \left| \int \langle F(x), G(x)\rangle\,dx \right|, \|F\|_{L^p(B)} \leq 1 \right\}$$

This means that $L^{p'}(B^*) \subset (L^p(B))^*$. Equality holds for $1 \leq p < \infty$ if B is reflexive, in particular when B is Hilbert.

The Marcinkiewicz-Zygmund theorem

If T is a bounded linear operator between $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$, and $1 \le r < \infty$ we consider the operator T^r defined on $L^p(\mathbb{R}^d, \ell^r)$ by

$$T^r((f_j)_j) = (Tf_j)_j$$

Strictly speaking, we consider T^r on finite sequences $\subset \ell^r$. If T^r is bounded from $L^p(\mathbb{R}^d, \ell^r)$ to $L^q(\mathbb{R}^q, \ell^r)$ that is

$$\|(\sum_j |Tf_j|^r)^{1/r}\|_q \le C \|(\sum_j |f_j|^r)^{1/r}\|_q$$

we say that T has an <u>ℓ^r-valued extension</u>.

It can be proved that if T is positive, meaning that $Tg \ge 0$ whenever $g \ge 0$, then this holds.

But for $r=2$ this holds for <u>all operators</u>

<u>Theorem (Marcinkiewicz-Zygmund)</u>. Let T be a bounded linear operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$. Then T has an ℓ^2-valued extension,

$$\|(\sum_j |Tf_j|^2)^{1/2}\|_q \le C_{p,q} \|T\| \|(\sum_j |f_j|^2)^{1/2}\|_p$$

For the proof we will use Gaussian sequences, that is, a sequence of i.i.d. random variables X_j in some probability space (Ω, P) with law given by the density $\underbrace{e^{-\pi x^2}}_{h(x)}$. We denote by b_r the L^r-norms

$$b_r = \left(\int |X_j|^r dP \right)^{1/r} = \left(\int |x|^r e^{-\pi x^2} dx \right)^{1/r}$$

<u>lemma</u>. If $(\lambda_j) \in \ell^2$ the series $\sum \lambda_j X_j$ converges in L^r, $r < \infty$, and

$$\| \sum_j \lambda_j X_j \|_r = b_r \left(\sum_j |\lambda_j|^2 \right)^{1/2}$$

<u>Proof of lemma</u>. Assume (λ_j) finite, $\sum_{j=1}^{N} |\lambda_j|^2 = 1$.

Let $X = \sum \lambda_j X_j$; let ρ an orthonormal transf. in $\mathbb{R}^d$ such that $\rho(\lambda_1, \lambda_2, \ldots, \lambda_N) = (1, 0, \ldots, 0)$. Then

$$P(X \in A) = \int_{\{\sum \lambda_j x_j \in A\}} h(x_1) h(x_2) \cdots h(x_N) \, dx =$$

$$= \int_{\{(\rho x)_1 \in A\}} e^{-\pi |x|^2} dx = \int_{\{x_1 \in A\}} e^{-\pi |x|^2} dx =$$

$$= \int_A e^{-\pi x_1^2} dx_1$$

That is X has the same law //

<u>Proof of theorem</u>. Assume first $q \leq p$. By lemma

$$\left| \left(\sum |T f_j(x)|^2 \right)^{1/2} \right|^q = b_q^{-q} \int | \sum_j T f_j(x) X_j |^q dP$$

~~And also~~ $= b_q^{-q} \int_\Omega | T \sum_j X_j f_j |^q dP$

$$\| (\sum_j |Tf_j|^2)^{1/2} \|_q^q = b_q^{-q} \int_{\mathbb{R}^d} (\int_\Omega |\sum_j Tf_j(x) X_j|^q dP) dx$$

$$= b_q^{-q} \int \| T(\sum_j X_j f_j) \|_q^q \, dP(\omega) \leq$$

$$\leq b_q^{-q} \|T\|^q \int \| \sum_j X_j f_j \|_p^q \, dP(\omega) \leq$$

$$\leq b_q^{-q} \|T\|^q \left(\int \| \sum_j X_j f_j \|_p^p \, dP(\omega) \right)^{q/p} =$$

$$= (b_p/b_q)^q \|T\|^q \| (\sum |f_j|^2)^{1/2} \|_q^q$$

If $p<q$, denote $s = q/p$. For every $u \geq 0$ with $\|u\|_{s'} \leq 1$ the operator $T_u f = u^{1/p} Tf$ satisfies

$$\|T_u f\|_p \leq \|Tf\|_q \leq \|T\| \, \|f\|_p$$

and we can apply the previous case to T_u $(q=p)$. Then

$$\| (\sum_j |Tf_j|^2)^{1/2} \|_q = \sup_u \left(\int (\sum |Tf_j|^2)^{p/2} u \right)^{1/p}$$

$$= \sup_u \| (\sum_j |T_u f_j|^2)^{1/2} \|_p \leq \|T\| \, \| (\sum_j |f_j|^2)^{1/2} \|_p \, . //$$

Vector-valued CZO operators

For Banach spaces A, B let $\mathcal{L}(A,B)$ be the space of bounded linear operators from A to B

Theorem (CZO for vector-valued). Assume that T is a bounded operator from $L^r(A)$ to $L^r(B)$ $1 < r \leq +\infty$. Assume too that there is a kernel K

$$K : \mathbb{R}^d \times \mathbb{R}^d \setminus \Delta \longrightarrow \mathcal{L}(A,B)$$

such that if $f \in L^r(A)$ has compact support and $x \notin \operatorname{spt} f$

$$Tf(x) = \int_{\mathbb{R}^d} K(x-y) \cdot f(y)\, dy$$

Assume finally that

$$\int_{|x-y| > 2|y-y'|} \|K(x,y) - K(x,y')\|_{\mathcal{L}(A,B)}\, dx \leq C$$

$$\int_{|x-y| > 2|x-x'|} \|K(x,y) - K(x',y)\|_{\mathcal{L}(A,B)}\, dx \leq C$$

Then T is bounded from $L^p(A)$ to $L^p(\mathbb{R}^d, B)$, $1 < p < \infty$ and is of weak type i.e.

$$\left|\{x : \|Tf(x)\|_B > \lambda\}\right| \leq \frac{C}{\lambda} \|f\|_{L^1(A)}$$

Proof is similar as in the scalar case (some technicalities involved when A is not reflexive).

We will apply this general setting to the following situation. We assume that $T_j f = k_j * f$ are convolution operators of CZ type, that is (case $q=2$)

$$|\widehat{k_j}(\xi)| = |m_j(\xi)| \le C$$

$$\int_{|x|>2|y|} |k_j(x-y) - k_j(x)|\, dx \le C \qquad y \in \mathbb{R}^d$$

$\boxed{\text{Ex 1}}$ In the first example we consider the operator

$$T(f) = (T_j f)_j$$

from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d, \ell^2)$. This will be the case if

$$\|T(f)\|_2^2 = \sum_j \|T_j f\|_2^2 = \sum_j \|m_j \hat{f}\|_2^2 = \int \Big(\sum_j |m_j(\xi)|^2\Big) |\hat{f}(\xi)|^2$$

is bounded by $\|f\|_2^2$, that is, if

$$\sum_j |m_j(\xi)|^2 \le C$$

The kernel of T is the function ~~with values~~

$$K(x) = (k_j(x))_j$$

So that the Hörmander condition reads

$$\int_{|x|>2y} \Big(\sum_j |k_j(x-y) - k_j(x)|^2\Big)^{1/2} dx \le C \qquad \forall y$$

Under these two assumptions, we thus have

$$\| (\sum_j |k_j * f|^2)^{1/2} \|_p \le C_p \| f \|_p , \quad f \in S$$

$$|\{x : (\sum_j |k_j * f|^2)^{1/2} > \lambda\}| \le \frac{C}{\lambda} \|f\|_{1} \quad f \in S$$

Ex 2. In the second example we consider the operator

$$\tilde{T}(f_j) = (T_j f_j)$$

from $L^2(\mathbb{R}^d, \ell^2)$ to $L^2(\mathbb{R}^d, \ell^2)$. This will be the case if

$$\|\tilde{T}(f_j)\|_2^2 = \sum_j \|T_j f_j\|_2^2 = \sum_j \|m_j \hat{f}_j\|_2^2 \le C \sum_j \|f_j\|_2^2$$

in particular if

$$\sup_j |m_j(\xi)| \le C$$

The kernel of $\tilde{T}$ is now the function that assigns to x the operator from ℓ^2 to ℓ^2

$$\tilde{K}(x)(\lambda_j) = (k_j(x)\lambda_j)_j$$

Consequently, $\|\tilde{k}(x)\|_{\mathcal{L}(\ell^2,\ell^2)} = \sup_j |k_j(x)|$.

Therefore the Hörmander condition reads:

$$\int_{|x|>2|y|} (\sup_j |k_j(x-y) - k_j(x)|)\, dy \le C \quad y \in \mathbb{R}^d$$

Under these two assumptions

$$\| (\sum_j |k_j * f_j|^2)^{1/2} \|_p \le C_p \| (\sum_j |f_j|^2)^{1/2} \|_p$$

$$|\{x : (\sum_j |k_j * f_j|^2)^{1/2} > \lambda\}| \le \frac{C}{\lambda} \| (\sum_j |f_j|^2)^{1/2} \|_1$$

This holds in particular if

$$\sup_j |m_j(\xi)| \le C \quad \forall j$$

$$|\nabla k_j(x)| \le \frac{C}{|x|^{d+1}} \quad \text{unif on } j$$

and of course if we choose $k_j = k \ \forall j$. In fact, if T is any scalar valued CZO, by the Marcinkiewicz-Zygmund theorem it has an extension to ℓ^2 and since conditions in theorem pg 95 hold it extends to $L^p(\mathbb{R}^d, \ell^2)$ to $L^p(\mathbb{R}^d, \ell^2)$.

Let us study in some detail H on the Schwarz class. First, note that the above holds as well for $f \in \mathcal{S}(\mathbb{R}^d)$: namely we write for $|x|$ big

$$\pi H f(x) = \int_{|x-y|<1} \frac{f(y)-f(x)}{x-y}\, dy + \int_{|x-y|>1} \frac{f(y)}{x-y}\, dy.$$

In the second term,

$$|x \frac{f(y)}{x-y}| \leq (1+|y|)|f(y)|.$$

In the second one, by Taylor expansion at y, $f(x)-f(y) = f'(y)(x-y)+R$, with $|R| \leq \frac{1}{2}|f''(z)||x-y|^2$ for some z between x and y; since $|z| > ||y|-1| > c|y|$ and f'' is rapidly decreasing we see that $R = O(|y|^{-2})$; altogether the first term is bounded by $|f'(y)| + |y|^{-2}$ uniformly in x. By dominated convergence we conclude that $\lim_{x\to\infty} xHf(x) = \frac{1}{\pi}\int f$. If $\int f = 0$, then we can substract $\frac{1}{x}$ from the kernel and get

$$Hf(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} f(y)\left(\frac{1}{x-y} - \frac{1}{x}\right) dy = \frac{1}{x} H(yf)(x),$$

so that $x^2 Hf(x)$ has limit $\frac{1}{\pi}\int yf$. Iterating we see that if all the moments $\int x^k f$ are zero, then $x^k Hf(x)$ has limit zero at ∞ for all k.

Now $D^{(n)}Hf = D^{(n)}(f * \frac{1}{\pi} p.v. \frac{1}{x}) = (D^{(n)} f) * \frac{1}{\pi} p.v. \frac{1}{x} = H(D^{(n)}f)$; on the other hand, integration by parts shows that all moments of $D^{(n)}f$ are zero if $n \geq 1$. It follows then that $x^k D^{(n)} Hf(x)$ has limit zero at infinity for all k and all $n \geq 1$. The only obstacle for Hf to be in $\mathcal{S}(\mathbb{R}^d)$ is $n = 0$ and this holds if and only if all moments of f are zero. Incidentally we can see this in a clear way on the Fourier transform side, because for $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$, the function $sign(\xi)\hat{f}(\xi)$ is in $\mathcal{S}(\mathbb{R}^d)$ if and only if $\hat{f}$ is flat at zero, that is, it has all derivatives zero at zero, which exactly says that the moments of f are zero.

3.8 Multipliers of $L^p(\mathbb{R}^d)$

We will see that the boundedness of the Hilbert transform has some important consequences. But first we explain some general facts about multipliers, and in doing so we place ourselves again in a general dimension $d \geq 1$.

We consider bounded linear operators $T : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d), 1 \leq p, q \leq \infty$ that commute with translations. A general theorem of Hormander (see Harmonic Analysis course) says that if p, q are both finite such an operator is given by convolution with a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d), Tf = f * u$ so that its action on $\varphi \in \mathcal{S}(\mathbb{R}^d)$ can be read $\widehat{T\varphi}(\xi) = \hat{\varphi}(\xi)m(\xi)$, with $m = \hat{u} \in \mathcal{S}'(\mathbb{R}^d)$.

We denote by $\mathcal{M}^{p,q}(\mathbb{R}^d)$ the space of all translation invariant operators from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$. A first point to remark is that we may assume that $p \leq q$:

100

Theorem 8. $\mathcal{M}^{p,q}(\mathbb{R}^d)$ reduces to zero if $p > q$.

Proof. The proof is similar to what was shown when seeing that the Fourier transform cannot be bounded from L^p to $L^{p'}$ if $p > 2$. Let us consider $f \in \mathcal{S}(\mathbb{R}^d)$ such that $Tf \neq 0$ and g a sum of N translates of $f, g(x) = \sum_{k=1}^N f(x - \lambda_k)$. If the λ_k are chosen spread enough we will have that $\|g\|_p = N^{\frac{1}{p}} \|f\|_p$. On the other hand, since T is translation invariant, $Tg(x) = \sum_{k=1}^N Tf(x - \lambda_k)$. But for an arbitrary non zero $h \in L^q(\mathbb{R}^d), q < \infty$,

$$\left(\int_{\mathbb{R}^d} |\sum_{k=1}^{N} h(x - \lambda_k)|^q \, dx, \right)^{\frac{1}{q}}$$

behaves like $N^{\frac{1}{q}}$ if the λ_k tend to infinity in a spread way (this is proved first for h with compact support). Thus we would have

$$N^{\frac{1}{q}} \leq C \|T\| N^{\frac{1}{p}},$$

for all N, and so $q \geq p$ if T is not zero. $\square$

We remind that in two cases we have already characterized the space $\mathcal{M}^{p,q}(\mathbb{R}^d)$. Namely, when $p = q = 1, \mathcal{M}^{p,q}(\mathbb{R}^d)$ is the space of finite complex Borel measures, that is, convolution with a finite measure is the general translation invariant operator in $L^1(\mathbb{R}^d)$. Also, when $p = q = 2$, we know that the translation invariant operators in $L^2(\mathbb{R})$ correspond exactly with the bounded multipliers m.

The case $p = q = \infty$ is exceptional. Of course convolution with a finite measure is an example, but there are translation invariant operators in $L^\infty(\mathbb{R}^d)$ that are not given by convolution. For example, let S be defined by

$$Sf = \lim_{R \to +\infty} \frac{1}{R} \int_0^R f(x) \, dx,$$

on the space of bounded periodic functions, and extend it to $L^\infty(\mathbb{R})$ using Hahn-Banach Theorem. Then T is a continuous linear operator onto the space of constants functions that commutes with translations and is not given by convolution because its action on test functions is zero.

So from now on we assume that $p \leq q, q > 1$ and leave the case $p = q = \infty$ aside. For such T, its transpose T^t is the operator $T^t : L^{q'}(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$ defined by

$$\langle Tf, g \rangle = \int_{\mathbb{R}^d} Tfg dx = \langle f, T^t g \rangle = \int_{\mathbb{R}^d} fT^t g, f \in L^p(\mathbb{R}^d), g \in L^{q'}(\mathbb{R}^d).$$

Then T^t is translation invariant and $T \mapsto T^t$ establishes a bijection between $\mathcal{M}^{p,q}(\mathbb{R}^d)$ and $\mathcal{M}^{q',p'}(\mathbb{R}^d)$. In general, if T is given by a kernel K

101

$$Tf(x) = \int_{\mathbb{R}^d} f(y)K(x,y)\,dy,$$

then

$$T^t g(y) = \int_{\mathbb{R}^d} g(x)K(x,y)\,dx,$$

that is, T^t is given by the kernel $K^t(x,y) = K(y,x)$. When it is translation invariant then $K(x,y) = K(x-y)$. In terms of the multiplier, if $m(\xi)$ is the multiplier of T, then $m(-\xi)$ is the multiplier of T^t. Since obviously all L^p spaces are stable by the reflection operator, we see that in fact $\mathcal{M}^{p,q}(\mathbb{R}^d)$ and $\mathcal{M}^{q',p'}(\mathbb{R}^d)$ are equal. Besides, by the Riesz interpolation theorem, we know that then T will be bounded too from $L^r(\mathbb{R}^d)$ to $L^s(\mathbb{R}^d)$ if $r = r_t, s = s_t$ are the required convex combinations. When $p = q$, we call $\mathcal{M}^p(\mathbb{R}^d)$ the space of all translation invariant operators from $L^p(\mathbb{R}^d)$ to itself, $1 < p < \infty$. Thus $\mathcal{M}^{p'}(\mathbb{R}^d) = \mathcal{M}^p(\mathbb{R}^d)$ and we can assume $1 < p \le 2$. By the Riesz interpolation theorem just mentioned, it follows that for $1 \le p < q \le 2$,

$$\mathcal{M}^p(\mathbb{R}^d) \subset \mathcal{M}^q(\mathbb{R}^d) \subset \mathcal{M}^2(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$$

.

Thus all multipliers m of $L^p(\mathbb{R}^d)$ are also multipliers of $L^2(\mathbb{R}^d)$ and so they are bounded. For $p = 1, \mathcal{M}^1(\mathbb{R}^d)$, is the smallest, it consists in convolution with measures μ, that is bounded in L^p, and has multiplier $m = \widehat{\mu}$, a bounded function

3.9 Some multipliers of $L^p(\mathbb{R})$

We have seen that H is a multiplier of $L^p(\mathbb{R}), 1 < p < \infty$, and the same holds for the Cauchy transform, the projection onto the holomorphic subspace. From this it is immediate to see that any filter $S_{a,b}$ in frequency, that is, the operator with multiplier

$$m(\xi) = 1_{[a,b]}(\xi),$$

is also an L^p multiplier. If M_a denotes the operation of multiplication with $e^{2\pi i a x}$, which is obviously bounded and that corresponds to the translation τ_a in frequency, just notice that

$$S_{a,b} = \frac{i}{2}(M_a H M_{-a} - M_b H M_{-b}).$$

Moreover we get from this that their operator norm is uniformly bounded in a, b,

$$\|S_{a,b}f\|_p \le C_p \|f\|_p.$$

Let us consider $S_R = S_{-R,R}$ which for $f \in L^2(\mathbb{R})$ is given by

An example by Hörmander

We have shown that $\mathcal{M}^p(\mathbb{R}^d) \subset \mathcal{M}^2(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$ that is, all L^p-multipliers are bounded functions.

This inclusion is strict, that is, for every $p \neq 2$ there are bounded functions which are not L^p-multipliers. We will see that there are $m \in L^\infty(\mathbb{R}^d)$, $|m(\xi)| = 1$, which are not L^p-multipliers, ~~that is, there exists~~

$m \in \mathcal{M}^p(\mathbb{R}^d)$ means that the operator defined on $S(\mathbb{R}^d)$ by

$$\widehat{T\varphi}(\xi) = m(\xi)\hat{\varphi}(\xi), \quad \varphi \in S(\mathbb{R}^d)$$

satisfies $\|T\varphi\|_p \le C_p \|\varphi\|_p$. Then this operator extends to the whole of $L^p(\mathbb{R}^d)$. For $1 \le p \le 2$, we know that the Fourier transform maps $L^p(\mathbb{R}^d)$ to $L^{p'}(\mathbb{R}^d)$, hence the extension will satisfy too

$$\widehat{T\varphi}(\xi) = m(\xi)\hat{\varphi}(\xi)$$

as functions on $L^{p'}(\mathbb{R}^d)$, that is a.e. ξ.

Theorem $m(\xi) = e^{i\lambda|\xi|^2}$, is not a multiplier of L^p, $p \neq 2$.

Proof. We first prove that if $\widehat{T_\lambda\varphi}(\xi) = e^{i\lambda|\xi|^2}\hat{\varphi}(\xi)$, then

$$\|T_\lambda\varphi\|_p \le C_p(\varphi)|\lambda|^{\frac{1}{p}-\frac{1}{2}}, \quad \varphi \in S(\mathbb{R}^d).$$

Of course we have $\|T_\lambda\varphi\|_2 = \|\varphi\|_2$. Using polar coordinates

$$T_\lambda\varphi(x) = \int e^{i(2\pi x\cdot\xi + \lambda|\xi|^2)}\hat{\varphi}(\xi)\,d\xi =$$

$$= \int_0^\infty \int_{|w|=1} e^{i(2\pi r\, x\cdot w + \lambda r^2)}\hat{\varphi}(rw)\, r^{d-1}\, dr\, d\sigma(w)$$

We integrate by parts in r. First notice that

$$\left| \int_0^R e^{i(2ar+\lambda r^2)} \, dr \right| \le C |\lambda|^{-1/2}$$

with C independent of R, a, t. Indeed, with the change of variable $x = \sqrt{\lambda}\,(r + \frac{a}{\lambda})$ it equals

$$\lambda^{-1/2} \, e^{-i\frac{a^2}{\lambda}} \int_{a/\sqrt{\lambda}}^{R\sqrt{\lambda}+a/\sqrt{\lambda}} e^{ix^2} \, dx$$

and we ~~the~~ know that $\int_{-\infty}^{+\infty} e^{ix^2} \, dx$ is convergent. Hence we have, integrating by parts

$$|T_\lambda \varphi(x)| \le C|\lambda|^{-1/2} \int \int_0^{\infty} \left| \frac{\partial \hat{\varphi}}{\partial r} + (d-1) \frac{\hat{\varphi}}{r} \right| r^{d-1} \, dr \, d\sigma(\omega) = C(\varphi) |\lambda|^{-1/2}$$

Then

$$\|T_\lambda \varphi\|_p^p = \int |T_\lambda \varphi(x)|^p \, dx \le C_p(\varphi) \, |\lambda|^{-\frac{1}{2}(p-2)} \int |T_\lambda \varphi|^2 dx =$$

$$= C_p(\varphi) \, |\lambda|^{1-\frac{p}{2}}$$

Suppose that $e^{-i\lambda|\xi|^2}$ were a multiplier for L^p. Then its norm as multiplier would be independent of λ that is

$$\|T_\lambda \varphi\|_p \le C_p \|\varphi\|_p$$

which is the same as (using $T_\lambda^{-1} = T_{-\lambda}$)

$$\|\varphi\|_p \le C_p \|T_\lambda \varphi\|_p$$

Hence we will have

$$\|\varphi\|_p \le C_p(\varphi)\, |\lambda|^{\frac{1}{p}-\frac{1}{2}}$$

and we reach a contradiction if $p>2$ (which we may assume) //

The estimate we obtained for $\int_0^R e^{i(2ar+\lambda r^2)}\,dr$

is a particular case of the Van-der Corput lemma on estimates for oscillatory integrals, which is worth-mentioning

Van der Corput lemma

(a) Assume that $|\phi^{(k)}(x)| \ge 1$ for $x \in (a,b)$ with $k \ge 2$ or that $|\phi'(x)| \ge 1$ and ϕ' is monotonic. Then

$$\left| \int_a^b e^{i\lambda\phi(x)}\,dx \right| \le C_k \lambda^{-1/k}$$

with C_k independent of a, b, λ, ϕ

(b) Under the same hypothesis,

$$\left| \int_a^b e^{i\lambda\phi(x)}\,\psi(x)\,dx \right| \le C_k \lambda^{-1/k} \left[|\psi(b)| + \int_a^b |\psi'(x)|\,dx \right]$$

(see proof in [Stein, Harmonic Analysis

105

$$Tf(x) = \int_{\mathbb{R}^d} f(y)K(x,y)\,dy,$$

then

$$T^t g(y) = \int_{\mathbb{R}^d} g(x)K(x,y)\,dx,$$

that is, T^t is given by the kernel $K^t(x,y) = K(y,x)$. When it is translation invariant then $K(x,y) = K(x-y)$. In terms of the multiplier, if $m(\xi)$ is the multiplier of T, then $m(-\xi)$ is the multiplier of T^t. Since obviously all L^p spaces are stable by the reflection operator, we see that in fact $\mathcal{M}^{p,q}(\mathbb{R}^d)$ and $\mathcal{M}^{q',p'}(\mathbb{R}^d)$ are equal. Besides, by the Riesz interpolation theorem, we know that then T will be bounded too from $L^r(\mathbb{R}^d)$ to $L^s(\mathbb{R}^d)$ if $r = r_t, s = s_t$ are the required convex combinations. When $p = q$, we call $\mathcal{M}^p(\mathbb{R}^d)$ the space of all translation invariant operators from $L^p(\mathbb{R}^d)$ to itself, $1 < p < \infty$. Thus $\mathcal{M}^{p'}(\mathbb{R}^d) = \mathcal{M}^p(\mathbb{R}^d)$ and we can assume $1 < p \leq 2$. By the Riesz interpolation theorem just mentioned, it follows that for $1 \leq p < q \leq 2$,

$$\mathcal{M}^p(\mathbb{R}^d)\mathcal{M}^q(\mathbb{R}^d)\mathcal{M}^2(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$$

.

Thus all multipliers m of $L^p(\mathbb{R}^d)$ are also multipliers of $L^2(\mathbb{R}^d)$ and so they are bounded. For $p = 1, \mathcal{M}^1(\mathbb{R}^d)$, is the smallest, it consists in convolution with measures μ, that is bounded in L^p, and has multiplier $m = \widehat{\mu}$, a bounded function

3.9 Some multipliers of $L^p(\mathbb{R})$

We have seen that H is a multiplier of $L^p(\mathbb{R}), 1 < p < \infty$, and the same holds for the Cauchy transform, the projection onto the holomorphic subspace. From this it is immediate to see that any filter $S_{a,b}$ in frequency, that is, the operator with multiplier

$$m(\xi) = 1_{[a,b]}(\xi),$$

is also an L^p multiplier. If M_a denotes the operation of multiplication with $e^{2\pi i a x}$, which is obviously bounded and that corresponds to the translation τ_a in frequency, just notice that

$$S_{a,b} = \frac{i}{2}(M_a H M_{-a} - M_b H M_{-b}).$$

Moreover we get from this that their operator norm is uniformly bounded in a, b,

$$\|S_{a,b} f\|_p \leq C_p \|f\|_p.$$

Let us consider $S_R = S_{-R,R}$ which for $f \in L^2(\mathbb{R})$ is given by

106

$$S_R f(x) = \int_{-R}^{R} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi.$$

We know that $S_R f \to f$ in $L^2(\mathbb{R})$ if $f \in L^2(\mathbb{R})$ by the very definition of the Fourier transform in L^2. If we can exhibit a dense subspace of $L^p(\mathbb{R})$ in which $S_R f \to f$ then by the uniform boundedness principle this would hold for all f. That space is simply the subspace of $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\hat{\varphi}$ compactly supported. Obviously $S_R\varphi = \varphi$ for R big enough. To see that it is dense it suffices to approximate $f \in \mathcal{S}(\mathbb{R}^d)$ in L^p-norm by such functions. If Ψ is C^∞ and has compact support and equals 1 at 0, then $\hat{\Psi}_t(x) = t^{-d}\hat{\Psi}(x/t)$ is an approximation of the identity so $f * \hat{\Psi}_t \to f$ in L^p, and has Fourier transform equal to $\hat{f}(\xi)\Psi(t\xi)$, with compact support.

We have thus shown that $S_R f \to f$ in $L^p(\mathbb{R}), 1 < p < \infty$. An analogous argument would show that for $p = 1$ the convergence is in measure. For $1 \leq p \leq 2, \hat{f} \in L^{p'}(R)$ and $S_R f$ is given by the above expression, but this is not the case for $p > 2$. Along the same lines, applying H to $f = \lim_R S_R f$ we see that

$$Hf(x) = -i \lim_{R\to\infty} \int_{-R}^{R} sign(\xi)\hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi,\, f \in L^p(\mathbb{R}), 1 < p \leq 2.$$

Theorem 9. Assume that m is a function of bounded variation in $\mathbb{R}$, that is

$$\sup \sum_{j=1}^{N} |m(x_j) - m(x_{j-1})| \leq C < +\infty,$$

for all points $x_0 < x_1 < \cdots < x_N$. Then m is a multiplier for $L^p(\mathbb{R}^d), 1 <$ $p < +\infty$.

Proof. We may assume that m is up to an additive constant the distribution function of a finite measure,

$$m(\xi) = c + \int_{-\infty}^{\xi} d\mu(t),$$

that we may write in a compact form

$$m = c + \int_{-\infty}^{+\infty} 1(t, +\infty) d\mu(t).$$

Then the operator T_m with symbol m is

$$T_m = cI + \int_{-\infty}^{+\infty} S_{(t,+\infty)} d\mu(t),$$

that is, an infinite linear combination with summable coefficients of the $S_{(t,+\infty)}$ which are uniformly bounded in L^p by C_p, and so T_m is bounded in L^p. $\square$

It is easy to obtain examples of multipliers in general dimension from multipliers in $d = 1$. Indeed, if $m(\xi_1)$ is a multiplier in $L^p(\mathbb{R})$, then this same function viewed as independent from $\xi_2, \cdots, \xi_d$ is a multiplier in $L^p(\mathbb{R}^d)$, with the same norm, by Fubini's theorem. On the other hand, if m is a multiplier in $L^p(\mathbb{R}^d)$, then $m(\xi + a), m(\lambda\xi), m(A\xi)$ with $a \in \mathbb{R}^d, \lambda > 0$, and A an unitary matrix are also multiplier with the same norm. Combining both things, we see that the characteristic function of a half-space is a multiplier for $L^p(\mathbb{R}^d), 1 < p < \infty$. The characteristic function of a convex polyhedra with N faces is the product of the N characteristic functions of half-spaces, and so it is a multiplier. This implies, in a similar way as we saw before

Theorem 10. The characteristic function of a convex polyhedra U is a multiplier of $L^p(\mathbb{R}^d), 1 < p < \infty$. In particular, $\lim_{\lambda \to \infty} S_{\lambda U} f = f$ in L^p.

The situation is quite different for other convex bodies. For instance, a celebrated theorem of C. Fefferman establishes that the characteristic function of a ball is a multiplier only for $p = 2$.

Littlewood - Paley theory

The motivation for the LP theory is Plancherel's theorem: $\|f\|_2 = \|\hat{f}\|_2$. This implies in particular

- The norm of f in L^2 only depends on $|\hat{f}|$
- If $f, g \in L^2(\mathbb{R}^d)$ and $\operatorname{spt} \hat{f}, \operatorname{spt} \hat{g}$ are disjoint

then

$$\|f\|_2^2 + \|g\|_2^2 = \|f+g\|_2^2 = \|\hat{f}\|_2^2 + \|\hat{g}\|_2^2$$

We ask ourselves what happens in L^p, $p \neq 2$. The first thing we must point out is that for $1 \leq p \leq 2$, $\hat{f} \in L^{p'}(\mathbb{R}^d)$ if $f \in L^p(\mathbb{R}^d)$. For $p > 2$, the Fourier transform of $L^p(\mathbb{R}^d)$ is a space of tempered distributions.

The first question is then: is it possible to control $\|f\|_p$ in terms of just the size $|\hat{f}|$ of $\hat{f}$? To be precise: for $1 \leq p \leq 2$ this means

- Is it possible to control $\|f\|_p$ by $|\hat{f}(\xi)|$, $f \in L^p$, $1 \leq p \leq 2$

- For $\varphi \in S(\mathbb{R}^d)$ is it possible to control $\|\varphi\|_p$ by $|\hat{\varphi}(\xi)|$

The answer is no. Indeed, we saw e.g. that $e^{i|\xi|^2}$ is not an L^p-multiplier if $p \neq 2$, and Fefferman theorem states that $\mathbb{1}_B$ is not an L^p-multiplier if $p \neq 2$. This means in particular that there exists for $1 < p < 2$ two functions $f, g \in L^2(\mathbb{R}^d)$ such that

$$\hat{f}(\xi) = e^{i|\xi|^2}\hat{g}(\xi)$$

and so $|\hat{f}| = |\hat{g}|$, $g \in L^p(\mathbb{R}^d)$ and $f \notin L^p(\mathbb{R}^d)$.

This is so because if it were the case that for all $g \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ the function f defined by $\hat{f}(\xi) = e^{i|\xi|^2}\hat{g}(\xi)$ is in $L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, the closed graph theorem applied to the map

$$g \longrightarrow f$$

would imply automatically that $\|f\|_p \le C\|g\|_p$ and so $e^{i|\xi|^2}$ would be a multiplier.

Thus we know that the pair g, f exists but no explicit example is known (at least by me).

So $\|\varphi\|_p$ does not depend only on $|\hat{\varphi}|$ if $p \neq 2$. We cannot multiply $\hat{\varphi}$ by m with $|m| = 1$ ~~with~~ and stay in L^p with controlled norm. The Littlewood-Paley offers a substitute for Plancherel's theorem and is based on a dyadic decomposition of the frequencies ξ.

There is an important technical difference between $d=1$ and $d>1$ motivated by the fact that characteristic functions of annuli are not L^p-multipliers if $d>1$. We explain first the result ~~...~~ for dyadic pieces

We consider the symmetric dyadic intervals

$$\Delta_j = (-2^{j+1}, -2^j] \cup [2^j, 2^{j+1}), \; j \in \mathbb{Z}$$

and call Δ a dyadic ~~interval~~ piece of $\mathbb{R}^d$ if it is the product of d dyadic intervals in $\mathbb{R}$. Let us call $\mathcal{D}$ the family of all dyadic intervals in $\mathbb{R}^d$ (note that this is not the same family that was used in the stopping time argument for the proof of the Calderon-Zygmund lemma). Each Δ is a union of 2^d rectangles.

For each $\Delta \in \mathcal{D}$ we consider the partial sum operator

$$\widehat{S_\Delta f} = \mathbb{1}_\Delta \hat{f}$$

that we know is an L^p-multiplier with constant independent of Δ and form the <u>square function</u>

$$Sf = \Big(\sum_\Delta |S_\Delta f|^2 \Big)^{1/2}$$

Plancherel's theorem says that $\|Sf\|_2 = \|f\|_2$.

<u>Littlewood-Paley theorem</u> For $1<p<\infty$ there are constants A_p, B_p such that $A_p\|f\|_p \le \|Sf\|_p \le B_p\|f\|_p$

In particular, if m is bounded and constant in each dyadic interval, then m is a multiplier for L^p, $1<p<\infty$.

In $d>1$, this does not hold replacing the Δ by the dyadic shells

$$D_j = \{ \xi : 2^j \le |\xi| < 2^{j+1} \}, \; j \in \mathbb{Z}$$

but the proof will show that it holds for a smooth version of this decomposition.

The proof will be achieved in several steps. First we introduce some notation. Generally speaking, we want to break $\hat{f}$ into dyadic pieces, that is, ψ would be a function say supported in $[1/2, 5/2]$ equal to 1 in $[1,2]$ and would consider the pieces $\psi(2^j \xi)\hat{f}(\xi)$.

This leads us to the following setting: we have a fixed function $k \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ ~~and~~ ~~consider~~ such that $k * f$ is a CZO and consider

$$k_j(x) = 2^{jd} k(2^j x), \quad j \in \mathbb{Z}$$

so that $\hat{k}_j(\xi) = \hat{k}(2^{-j}\xi)$, and consider the operator T

$$T(f) = (k_j * f)_j$$

from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d, \ell^2)$, and the square function ~~We also consider the~~

$$\left(\sum_j |k_j * f|^2\right)^{1/2} = S(f)(x)$$

We also consider the continuous version, namely

$$k_t(x) = t^{-d} k(x/t), \quad t > 0$$

and the operator G

$$G(f) = (k_t * f)_t$$

from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d, H)$ where H is the Hilbert space $L^2(0,\infty; \frac{dt}{t})$, and form the square function

$$\left(\int_0^\infty |(k_t * f)(x)|^2 \frac{dt}{t}\right)^{1/2} = G(f)(x)$$

(note that the Riemann sums corresponding to dyadic intervals $t \approx 2^{-j}$ of $G(f)(x)$ is $S(f)(x)$).

The following lemma isolates properties of k so that the two basic assumptions of the CZO theorem hold

$$\sum_j |\hat{k}_j(\xi)|^2 = \sum_j |\hat{k}(2^{-j}\xi)|^2 \leq C$$

$$\int_{|x|>2|y|} \Big(\sum_j |k_j(x-y) - k_j(x)|^2 \Big)^{1/2} dx \leq C$$

in the discrete case, and

$$\int_0^\infty |\hat{k}(t\xi)|^2 \frac{dt}{t} \leq C$$

$$\int_{|x|>2|y|} \Big(\int_0^\infty |k_t(x-y) - k_t(x)|^2 \frac{dt}{t} \Big)^{1/2} dx \leq C$$

in the continuous case.

<u>Lemma</u>. For the above to hold it is enough that $k \in C^1(\mathbb{R}^d)$, $k \in L^1(\mathbb{R}^d)$, $\hat{k}(0) = \int k(x)\,dx = 0$, and that for some $\alpha > 0$

$$|k(x)| \leq \frac{C}{(1+|x|)^{d+\alpha}}, \quad |\nabla k(x)| \leq \frac{C}{(1+|x|)^{d+\alpha+1}}$$

and hence in this case

$$\|S(f)\|_p \le C_p \|f\|_p, \quad \|G(f)\|_p \le C_p \|f\|_p, \quad 1<p<\infty, f(\cdot)$$

$$|\{x : |S(f)| > \lambda\}| \le \frac{C}{\lambda}\|f\|_1; \quad |\{x : |G(f)| > \lambda\}| \le \frac{C}{\lambda}\|f\|_1$$

<u>Proof of the lemma</u>. We first obtain pointwise estimates for $\hat{k}$.

$$\hat{k}(\xi) = \int_{\mathbb{R}^d} k(x) e^{-2\pi i x\xi} dx = \int k(x)\{e^{-2\pi i x \xi} - 1\} dx$$

For $|\xi|$ small, say $|\xi| \le 1$,

$$|\hat{k}(\xi)| \lesssim \int_{|x|\le|\xi|^{-\beta}} |k(x)|\,|x|\,|\xi|\, dx + \int_{|x|\ge|\xi|^{-\beta}} |x|^{-d-\alpha} dx = O(|\xi|^{\frac{\alpha}{1+\alpha}})$$

(choosing $\beta = 1/(1+\alpha)$) For $|\xi|$ large we use the fact that $|\xi|\hat{k}(\xi)$ ~~...~~ is the Fourier transform of ~~...~~ $\nabla k(x)$, so ~~choosing ...~~

$$|\xi|\,|\hat{k}(\xi)| \le \int |\nabla k(x)|\, dx$$

With this we can ~~prove~~ estimate (assuming by homogeneity $|\xi| = 1$)

$$\int_0^\infty |\hat{k}(t\xi)|^2 \frac{dt}{t} \lesssim \int_0^1 t^{\frac{2\alpha}{1+\alpha} - 1} dt + \int_1^\infty t^{-1-2} dt < +\infty$$

and analogously the discrete version. For the Hormander condition (continuous version), we may assume again by homogeneity that $|y| = 1$

$$\int_{|x|>2} \left(\int_0^\infty |k_t(x-y) - k_t(x)|^2 \frac{dt}{t} \right)^{1/2} dx \le$$

$$= \int_{|x|\geq 2} \left(\int_0^\infty \left| k\left(\frac{x}{t} - \frac{y}{t}\right) - k\left(\frac{x}{t}\right) \right|^2 t^{-2d-1} \, dt \right)^{1/2} dx = (*)$$

By the mean value theorem, $(|y| = 1)$

$$\left| k\left(\frac{x}{t} - \frac{y}{t}\right) - k\left(\frac{x}{t}\right) \right| \leq C \frac{1/t}{\left(1 + \left|\frac{x}{t}\right|\right)^{d+\alpha+1}} = C \frac{t^{d+\alpha}}{(t+|x|)^{d+\alpha+1}}$$

and so the t integral is bounded by

$$\int_0^\infty \frac{t^{2\alpha-1}}{(t+|x|)^{2d+2\alpha+2}} \, dt =$$

$$= |x|^{-2d-2} \int_0^\infty \frac{t^{2\alpha-1} \, dt}{(1+t)^{2d+2\alpha+2}} = O\left(|x|^{-2d-2}\right)$$

and we get

$$(*) \leq \int_{|x|\geq 2} |x|^{-d-1} \, dx = O(1).$$

Analogously the discrete version //

Theorem. Let $\psi \in S(\mathbb{R}^d)$ in the Schwartz class with $\int \psi \, dx = 0$. Let S_j be the operator defined by

$$\widehat{S_j f}(\xi) = \psi(2^{-j}\xi)\hat{f}(\xi), \quad S_j f = (k_j * f)(\xi),$$

$k_j(x) = 2^{jd} k(2^j x)$, $\hat{k} = \psi$. Then

$$\| (\sum_j |S_j f|^2)^{1/2} \|_p \le C_p \|f\|_p, \quad 1 < p < \infty$$

Also,

$$\left\| \left(\int_0^\infty |(k_t * f)(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_p \le C_p \|f\|_p.$$

Moreover, if

$$\sum_j |\psi(2^{-j}\xi)|^2 = C \qquad \xi \neq 0$$

then the reverse inequality holds for Sf

$$C_p' \|f\|_p \le \| (\sum_j |S_j f|^2)^{1/2} \|_p$$

Analogously, if

$$\int_0^\infty |\hat{k}(t\xi)|^2 \frac{dt}{t} = C \qquad \text{a.e. } \xi$$

the reverse inequality holds for $G(f)$

$$C_p' \|f\|_p \le \| (\int_0^\infty |(k_t * f)(x)|^2 \frac{dt}{t})^{1/2} \|_p$$

Proof. It is clear that the lemma can be applied to ψ, so that the direct inequalities hold. For the reverse inequalities we first observe that if

$$\sum_j |\psi(2^{-j}\xi)|^2 = C$$

then

$$\|(\sum_j |S_j f|^2)^{1/2}\|_2^2 = C\,\|f\|_2^2$$

Indeed, the left hand side is

$$\sum_j \|S_j f\|_2^2 = \sum_j \|\psi(2^{-j}\xi)\hat{f}\|_2^2 = \|(\sum_j |\psi(2^{-j}\xi)|^2)\hat{f}\|_2^2$$

If we polarize the above we get

$$\int_{\mathbb{R}^d} (\sum_j S_j f\, \overline{S_j g})\, dx = C\int_{\mathbb{R}^d} f\,\bar{g}\, dx \quad , f, g \in S(\mathbb{R}^d)$$

and then we can argue by duality:

$$\|f\|_p = \sup\{|\int f\bar{g}|, \|g\|_{p'} \le 1\} = C\sup\{|\int \sum_j S_j f\,\overline{S_j g}| : \|g\|_{p'} \le 1\} \le$$

$$\le C\sup\{\|(\sum_j |S_j f|^2)^{1/2}\|_p \|(\sum_j |S_j g|^2)^{1/2}\|_{p'} ; \|g\|_{p'} \le 1\} \le$$

$$\le C\,C_{p'}\,\|(\sum_j |S_j f|^2)^{1/2}\|_p$$

In an analogous way, if $\displaystyle\int_0^\infty |\hat{k}(t\xi)|^2 \frac{dt}{t} = C$ a.e. then

$$\left\| \left(\int_0^\infty |k_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_2^2 = C \, \|f\|_2^2$$

and we proceed in a similar way. //

~~Our aim is now to prove the Littlewood-Paley theorem using dyadic pieces instead of the "smooth pieces" $\psi(2^{-j}\xi)\hat{f}(\xi)$.~~

An easy way to construct $\psi \in S(\mathbb{R}^d)$ satisfying the hypothesis of the theorem is as follows: let $\varphi \in S(\mathbb{R}^d)$ $\varphi \geq 0$, radial, decreasing, with $\varphi(\xi) = 1$ for $|\xi| \leq 1/2$ and $\varphi(\xi) = 0$ for $|\xi| \geq 1$, and define

$$\psi^2(\xi) = \varphi\left(\frac{\xi}{2}\right) - \varphi(\xi)$$

Our aim is now the proof of the Littlewood-Paley theorem using dyadic pieces instead of the smooth pieces $\psi(2^{-j}\xi)\hat{f}(\xi)$. We carry out the proof only in dimension $d=1$.

The result will be based on the smooth case. Let $\psi \in S(\mathbb{R})$, $1 \geq \psi \geq 0$, supported in $\frac{1}{2} \leq |x| \leq 4$ and equal to 1 on $1 \leq |x| \leq 2$. We consider as above

$$\widehat{S_j f}(\xi) = \psi(2^{-j}\xi)\hat{f}(\xi)$$

We denote by $\Delta_j f$ the dyadic sum operators, that is, those corresponding to $\mathbb{1}_{[1,2]}$

$$\widehat{\Delta_j f}(\xi) = \mathbb{1}_{[1,2]}(2^{-j}\xi)\hat{f}(\xi)$$

Then we have $\Delta_j S_j = \Delta_j$.

<u>Lemma</u>. Let I_j be arbitrary intervals and

$\widehat{T_j f}(\xi) = \mathbb{1}_{I_j}(\xi)\hat{f}(\xi)$. Then

$$\Big\| \Big(\sum_j |T_j f_j|^r \Big)^{1/r} \Big\|_p \leq C_{p,r} \Big\| \Big(\sum_j |f_j|^r \Big)^{1/r} \Big\|_p$$

<u>Proof of the lemma</u> If $I_j = (a_j, b_j)$ then

$$T_j f_j = \frac{i}{2}\left(M_{a_j} H M_{-a_j} f_j - M_{b_j} H M_{-b_j} f_j \right)$$

where M denotes multiplication operators by characters. Then the result follows from

$$\Big\| \Big(\sum_j |Hg_j|^r \Big)^{1/r} \Big\|_p \leq C_{p,r} \Big\| \Big(\sum_j |g_j|^r \Big)^{1/r} \Big\|_p$$

that we saw it holds for a general CZO operator. //

<u>Proof of the Littlewood-Paley theorem</u>. Using $\Delta_j S_j = \Delta_j$

$$\|(\sum_j |\Delta_j f|^2)^{1/2}\|_p = \|(\sum_j |\Delta_j S_j f|^2)^{1/2}\|_p$$

is by the lemma just proved applied to $f_j = S_j f$ $(r=2)$

$$\leq C_p \|(\sum_j |S_j f|^2)^{1/2}\|_p$$

and this is in turn bounded by $C\|f\|_p$ by the smooth version proved before.

The reverse inequality follows as above by polarization and duality, because the Parseval's theorem tell us that

$$\|(\sum_j |\Delta_j f|^2)^{1/2}\|_2 = \|f\|_2 \quad . //$$

Relation of Littlewood-Paley theory with wavelets

In case $\int_0^\infty |\hat{k}(t\xi)|^2 \frac{dt}{t} = C$ we have ~~used~~ that

$$(*) \quad \int_{\mathbb{R}^d}\int_0^\infty |k_t * f(x)|^2 \frac{dt}{t}\, dx = K \int_{\mathbb{R}^d} |f(x)|^2\, dx \quad \left(\begin{array}{c}\text{trivial by}\\ \text{Parseval}\end{array}\right)$$

This equality can be interpreted in a more appealing way in the wavelet language.

To the "mother wavelet" k we associate the wavelets

$$k_{a,t}(x) = t^{-d/2}\, k\left(\frac{x-a}{t}\right) =$$

so that $\|k_{a,t}\|_2 = \|k\|_2$. The continuous wavelet transform of f is the map that associates to (a,t) the correlation

$$< f, k_{a,t} > = t^{-d/2} \int_{\mathbb{R}^d} f(x)\, \overline{k\left(\frac{x-a}{t}\right)}\, dx$$

We claim that the equality holds

$$C f = \iint < f, k_{a,t} > k_{a,t}\, da\, \frac{dt}{t^{d+1}}$$

Indeed, we first express $< f, k_{a,t} >$ differently

$$< f, k_{a,t} > = < \hat{f}, \hat{k}_{a,t} >$$

But $\widehat{k_{a,t}}(\xi) = t^{d/2} e^{-2\pi i \xi a}\, \hat{k}(t\xi)$, so

$$\langle f, k_{a,t}\rangle = t^{d/2}\int \hat{f}(\xi)\,\overline{\hat{k}(t\xi)}\, e^{2\pi i \xi a}\, d\xi$$

is the Fourier cotransform of $t^{d/2}$, evaluated at a.

We now compute

$$\int_{\mathbb{R}^d} \langle f, k_{a,t}\rangle k_{a,t}\, da = \int_{\mathbb{R}^d} \mathcal{F}^{-1}\left[\hat{f}(\cdot)\,\overline{\hat{k}(t\cdot)}\right](a)\, k\left(\frac{x-a}{t}\right)$$

By Parseval's formula, this equals

$$\int_{\mathbb{R}^d} \hat{f}(\xi)\,\overline{\hat{k}(t\xi)}\, \mathcal{F}_a^{-1}\left(k\left(\frac{x-a}{t}\right)\right)(\xi)$$

That is, we consider $k\left(\frac{x-a}{t}\right)$ as a function of a and compute its cotransform at ξ

$$\int_{\mathbb{R}^d} e^{2\pi i \xi a}\, k\left(\frac{x-a}{t}\right) da = e^{2\pi i \xi x}\, \hat{k}(t\xi)\, t^d$$

Hence

$$\int_{\mathbb{R}^d} \langle f, k_{a,t}\rangle k_{a,t}\, da = t^d \int_{\mathbb{R}^d} \hat{f}(\xi)\,|\hat{k}(t\xi)|^2 e^{2\pi i \xi x}\, d\xi$$

and so

$$\int_{\mathbb{R}^d} \langle f, k_{a,t}\rangle k_{a,t}\, da\, \frac{dt}{t^{d+1}} = \left(\int_0^\infty \frac{|\hat{k}(t\xi)|^2}{t}\, dt\right) f(x)$$

Multiplying by $\overline{f(x)}$ and integrating in x we get

$$\int_{\mathbb{R}^d} |\langle f, k_{a,t}\rangle|^2\, \frac{da\, dt}{t^{d+1}} = K\|f\|_2^2 \quad \text{(Parseval's thm)}$$

To compare this with our previous setting in which we used the notation $k_t(x) = t^{-d}\, k(x/t)$, we have then

$$k_{a,t}(x) = t^{d/2}\, k_t(x-a), \text{ so}$$

$$\frac{1}{t^d} \left| \langle f, k_{a,t} \rangle \right|^2 = \left| \int f(x)\, k_t(x-a)\, dx \right|^2$$

If $\tilde{k}(x) = k(-x)$, this equals $\left| (f * \tilde{k}_t)(a) \right|^2$, so Parseval's theorem becomes

$$\iint |f * \tilde{k}_t(a)|^2 \, da \, \frac{dt}{t} = K \, \|f\|_2^2$$

Applications of Littlewood-Paley theory to multipliers ($d=1$)

$$\|f\|_p \sim \|(\sum_j |\Delta_j f|^2)^{1/2}\|_p$$

A first and obvious application of this fact is that if m equals a constant m_j on each Δ_j, and $\sup_j |m_j| < +\infty$ then m is an L^p multiplier.

$$\Delta_j T_m f = m_j \Delta_j f \rightarrow$$

$$(\sum |\Delta_j T_m f|^2)^{1/2} \leq (\sup_j |m_j|)(\sum_j |\Delta_j f|^2)^{1/2}$$

In general, for $m \in L^\infty(\mathbb{R})$, let us write $m_j = m \mathbb{1}_{\Delta_j}$ and T_{m_j} for the operator with multiplier m_j, so that

$$T_m = \sum_j T_{m_j}$$

We claim that m is a multiplier if and only if

$$\|(\sum_j |T_{m_j} f_j|^2)^{1/2}\|_p \leq C \|(\sum_j |f_j|^2)^{1/2}\|_p$$

holds for an arbitrary family of functions (f_j), $f_j \in L^2 \cap L^p$. Indeed, if this holds then, since $\Delta_j T_m = T_{m_j} \Delta_j$

$$\|T_m f\|_p \sim \|(\sum_j |\Delta_j T_m f|^2)^{1/2}\|_p = \|(\sum_j |T_{m_j} \underbrace{\Delta_j f}_{f_j}|^2)^{1/2}\|_p$$

$$\leq C \|(\sum_j |\Delta_j f|^2)^{1/2}\|_p \sim \|f\|_p$$

In the other direction, we need again the lemma in page 118:

<u>Lemma</u>. For arbitrary intervals I_j, if $\widehat{T_j f} = \mathbb{1}_{I_j} \hat{f}$ then

$$\left\| \left(\sum |T_j f_j|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_p$$

So

Applying this lemma to the Δ_j: $T_{m_j} f_j = \Delta_j T_m f$

$$\left\| \left(\sum_j |T_{m_j} f_j|^2 \right)^{1/2} \right\|_p = \left\| \left(\sum_j |\Delta_j T_m f_j|^2 \right)^{1/2} \right\|_p \leq$$

$$\leq C_p \left\| \left(\sum_j |T_m f_j|^2 \right)^{1/2} \right\|_p$$

and then the Marcinkiewicz - Zygmund theorem finishes.

—— x —— x ——

In the same spirit we will now "cut into pieces" the statement that a function of bounded variation is an L^p-multiplier

<u>Marcinkiewicz Multiplier theorem</u> Assume $m \in L^\infty(\mathbb{R})$ and that on each Δ_j has bounded variation V_j, with $V = \sup_j V_j < +\infty$. Then m is an L^p-multiplier.

<u>Proof</u>. In each dyadic interval $\Delta_j = (2^j, 2^{j+1})$ we write

$$m_j(\xi) = m(2^j) + \int_{2^j}^{\xi} dm(t) = m(2^j) + \int_{2^j}^{2^{j+1}} \mathbb{1}_{(t,+\infty)}(\xi)\, dm(t)$$

This means

$$T_{m_j} f = m(2^j)\Delta_j f + \int_{2^j}^{2^{j+1}} (S_t \Delta_j f) dm(t)$$

where S_t denotes the operator $\widehat{S_t f}(\xi) = 1\!\!1_{(t,+\infty)}(\xi) \hat{f}(\xi)$. Then using Schwarz inequality for $dm(t)$

$$|T_{m_j} f| \le \|m\|_\infty |\Delta_j f| + \left(\int_{2^j}^{2^{j+1}} |S_t \Delta_j f|^2 dm(t)\right)^{1/2} V_j^{1/2}$$

and hence

$$|T_{m_j} f|^2 \lesssim |\Delta_j f|^2 + \int_{2^j}^{2^{j+1}} |S_t \Delta_j f|^2 dm(t)$$

If we have an arbitrary family f_j the lemma gives

$$\|(\sum_j |\Delta_j f_j|^2)^{1/2}\|_p \le C_p \|(\sum_j |f_j|^2)^{1/2}\|_p$$

Next

$$\sum_j \int_{2^j}^{2^{j+1}} |S_t \Delta_j f_j|^2 dm(t)$$

can be written

$$\int_{\mathbb{R}} |S_{J(t)} f_t|^2 dm(t)$$

with the notation $J(t) = [t,\infty) \cap \Delta_j$, $f_t = f_j$ if Δ_j is the dyadic interval to which t belongs

We need at this point the continuous version of the lemma. The lemma is a statement about ℓ^2. Replacing ℓ^2 by $L^2(dm)$ we get

$$\Big\| \int_{\mathbb{R}} |S_{J(t)} f_t|^2 \, dm(t) \Big\|_p \leq C \Big\| \int_{\mathbb{R}} |f_t|^2 \, dm(t) \Big\|_p$$

But

$$\int_{\mathbb{R}} |f_t|^2 dm(t) = \sum_j v_j |f_j|^2 \leq C \sum |f_j|^2$$

and we are done.

Hörmander–Mikhlin multiplier theorem ($d \geq 1$)

We look again at some "easy" criteria for m to be a multiplier of L^p, of global character, and then we will cut into pieces.

The global criteria we use is the following: we consider the Sobolev space $L^2_\alpha(\mathbb{R}^d)$ consisting in those $g \in L^2(\mathbb{R}^d)$ such that

$$(1+|\xi|^2)^{\alpha/2}\, \hat{g}(\xi) \in L^2(\mathbb{R}^d)$$

When $\alpha = m \in \mathbb{N}$, this mean $D^\beta g \in L^2$ $|\beta| \leq m$.

It is trivial that if $g \in L^2_\alpha(\mathbb{R}^d)$ and $\alpha > \frac{d}{2}$ then $\hat{g} \in L^1(\mathbb{R}^d)$, in particular g is bounded, continuous. (This is because if

$$h(\xi) = (1+|\xi|^2)^{\alpha/2}\, \hat{g}(\xi) \in L^2(\mathbb{R}^d) \text{ then}$$

$$\int |\hat{f}(\xi)|\, d\xi \leq \left(\int |h(\xi)|^2 d\xi\right)^{1/2} \left(\int \frac{d\xi}{(1+|\xi|^2)^\alpha}\right)^{1/2}.$$

So, if $m \in L^2_\alpha(\mathbb{R}^d)$ and $\alpha > \frac{d}{2}$ then m is a multiplier, because $m = \hat{h}$, with $h \in L^1(\mathbb{R}^d)$, T_m is convolution by h, and this is bounded in all L^p-spaces.

Theorem. Assume that $m \in L^\infty(\mathbb{R}^d)$ is such that

$$\sup_j \| m(2^j\xi)\, \psi(\xi) \|_{L^2_\alpha} < +\infty$$

for some $\psi \in S(\mathbb{R}^d)$, $\operatorname{spt} \psi \subseteq \{ 1/2 \leq |\xi| \leq 2 \}$,

$$\sum_{j=-\infty}^{+\infty} |\psi(2^{-j}\xi)|^2 = 1, \quad \xi \neq 0$$

Then m is a multiplier in L^p.

<u>Proof</u>. As in the proof of the LP theorem, let S_j denote the operator $\widehat{S_j f}(\xi) = \psi(2^{-j}\xi)\hat{f}(\xi)$. For these operators we have

$$\|f\|_p \sim \|(\sum_j |S_j f|^2)^{1/2}\|_p \qquad 1 < p < +\infty$$

We write $m_j(\xi) = m(\xi)\psi(2^{-j}\xi)$, ~~and~~ as before, it is enough to prove that

$$(*) \qquad \|(\sum_j |T_{m_j} f_j|^2)^{1/2}\|_p \leq C \|(\sum_j |f_j|^2)^{1/2}\|_p$$

for an arbitrary sequence (f_j). Indeed (in $d>1$ we cannot use Δ_j but smooth versions of them), let $\tilde{\psi}$ another $C_c^\infty(\mathbb{R}^d)$ function supported in $\frac{1}{4} \leq |\xi| \leq 4$ and equal to one in $\frac{1}{2} \leq |\xi| \leq 2$ and let $\tilde{S}_j$ be the operator $\widehat{\tilde{S}_j f}(\xi) = \tilde{\psi}(2^{-j}\xi)\hat{f}(\xi)$. Then, using $S_j T_m f = T_{m_j}\tilde{S}_j$

$$\|T_m f\|_p \sim \|(\sum_j |S_j T_m f|^2)^{1/2}\|_p = \|(\sum_j |T_{m_j} \underbrace{\tilde{S}_j f}_{f_j}|^2)^{1/2}\|_p$$

$$\leq C \|(\sum_j |\tilde{S}_j f|^2)^{1/2}\|_p \leq C \|f\|_p$$

(we have used LP for the S_j's and for the $\tilde{S}_j$'s)

So we must prove $(*)$. Let us denote by $M_j(\xi) = m(2^j\xi)\psi(\xi)$ so that the hypothesis is

$$\sup_j \|M_j\|_{L^2_\alpha} < +\infty$$

and $M_j(2^{-j}\xi) = m(\xi)\psi(2^{-j}\xi)$.

Lemma. Assume $M \in L^2_\alpha$ with $\alpha > \frac{n}{2}$ and $\lambda > 0$ and let T_λ be the operator $\widehat{T_\lambda f}(\xi) = M(\lambda\xi)\hat{f}(\xi)$. Then for an arbitrary $u \geq 0$

$$\int_{\mathbb{R}^d} |T_\lambda f(x)|^2 u(x) \leq C \int |f(x)|^2 M^* u(x)\,dx$$

where $M^* u$ denotes the Hardy-Littlewood maximal function of u (the constant C is independent of λ).

Proof. If $\hat{K} = M$, by hypothesis $(1+|x|^2)^{\alpha/2} K(x) = R(x) \in L^2(\mathbb{R}^d)$. The kernel of T_λ is $\lambda^{-d} K(x/\lambda)$. Then

$$\int_{\mathbb{R}^d} |T_\lambda f(x)|^2 u(x)\,dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \frac{\lambda^{-d} R\left(\frac{x-y}{\lambda}\right)}{\left(1 + \left|\frac{x-y}{\lambda}\right|^2\right)^{\alpha/2}} f(y)\,dy \right|^2 u(x)\,dx$$

We apply Cauchy-Schwarz to the inner integral; it is dominated by

$$\left|\int \quad \right|^2 \leq \left(\int \frac{\lambda^{-d} |f(y)|^2}{\left(1 + \left|\frac{x-y}{\lambda}\right|^2\right)^{\alpha/2}}\,dy\right)\left(\int \lambda^{-d} R\left(\frac{x-y}{\lambda}\right)^2 dy\right) =$$

$$= \left(\int \frac{\lambda^{-d} |f(y)|^2\,dy}{\left(1 + \left|\frac{x-y}{\lambda}\right|^2\right)^{\alpha/2}}\right) \underbrace{\|R\|_2^2}_{= \|M\|_{L^2_\alpha}}.$$

So

$$\int_{\mathbb{R}^d} |T_\lambda f(x)|^2 u(x)\,dx \leq \|M\|_{L^2_\alpha} \iint \frac{\lambda^{-d} |f(y)|^2 u(x)}{\left(1 + \left|\frac{x-y}{\lambda}\right|^2\right)^{\alpha/2}}\,dx\,dy$$

But
$$\int_{\mathbb{R}^d} \frac{\lambda^{-d} u(x)}{\left(1+\left|\frac{x-y}{\lambda}\right|^2\right)^{d/2}} \leq C\, M^* u(y)$$

because $\chi(x) = \dfrac{1}{(1+|x|^2)^{d/2}}$ is radial, in $L^1(\mathbb{R}^d)$ and we already know that for those $\sup_\lambda |\chi_\lambda * u| \leq C M^* u$ //

The lemma applied to m_j and $\lambda = 2^{-j}$ gives
$$\int |T_{m_j} f_j(x)|^2 u(x)\, dx \leq C \int |f_j(x)|^2 M^* u(x)\, dx.$$
with C independent of j, by the main assumption.

Now we will use this for a convenient u. We may assume that $p>2$. Then
$$\| (\sum_j |T_{m_j} f_j|^2)^{1/2} \|_p^2 = \| \sum_j |T_{m_j} f_j|^2 \|_{p/2} =$$
$$= \int_{\mathbb{R}^d} (\sum_j |T_{m_j} f_j|^2(x))\, u(x)\, dx$$
for some $u \in L^{(p/2)'}(\mathbb{R}^d)$, $u \geq 0$, with $\|u\|_{(p/2)'} = 1$.

Using the above, this is in turn bounded by
$$\int_{\mathbb{R}^n} (\sum_j |f_j|^2)\, M^* u(x)\, dx \leq \| (\sum_j |f_j|^2) \|_{p/2} \, \| M^* u \|_{(p/2)'}$$
But M^* is bounded in $L^{(p/2)'}$ and so we get
$$\| (\sum_j |T_{m_j} f_j|^2)^{1/2} \|_p^2 \leq C \| \sum_j |f_j|^2 \|_{p/2} = C \| (\sum |f_j|^2)^{1/2} \|_p^2$$ //

Corollary (Hörmander-Mihlin multiplier theorem).

Let $n = \left[\frac{d}{2}\right] + 1$, m of class C^k in $\mathbb{R}^d \setminus \{0\}$ and assume that

$$\sup_R \left(\frac{1}{R^n} \int_{R < |\xi| < 2R} |D^\alpha m(\xi)|^2 d\xi \right)^{1/2} \leq C R^{-|\alpha|}$$

for all multiindexes α, $|\alpha| \leq n$. Then m is an L^p-multiplier, $1 < p < \infty$. This condition is satisfied in particular if

$$|D^\alpha m(\xi)| = O\left(|\xi|^{-|\alpha|}\right), \quad |\alpha| \leq n.$$

Proof The (first) assumption may be rewritten

$$\sup_R \left(\int_{1 < |\xi| < 2} |D^\alpha m(R\xi)|^2 d\xi \right)^{1/2} \leq C \qquad |\alpha| \leq n$$

Then, if φ is as above ($\varphi \in C_c^\infty(\mathbb{R}^d)$, $\operatorname{spt} \varphi \subset \{\frac{1}{2} \leq |\xi| \leq 3\}$ & $\sum_j |\varphi(2^{-j}\xi)|^2 = 1$),

$$|D^\alpha(m(2^j\xi)\varphi(\xi))| = \left|\sum_{|\gamma| \leq |\alpha|} C_{\alpha,\gamma} D^\gamma[m(2^j\xi)] D^{\alpha-\gamma}\varphi\right|$$

$$\leq C \sum_{|\gamma| \leq |\alpha|} |D^\gamma[m(2^j\xi)]|$$

This will in $L^2(\mathbb{R}^d)$ for $|\alpha| \leq n$, uniformly in j.

Since $n = \left[\frac{d}{2}\right] + 1 > \frac{d}{2}$, we are done. //

<u>Corollary</u>. If m is homogeneous of degree 0 and of class C^{n} in S^{d-1} with $n > [\frac{d}{2}]$, then m is an L^p-multiplier, $1 < p < \infty$.

We knew this before for $m \in C^\infty(S^{d-1})$.