L^2 boundedness without Fourier transform

In the CZO theory, a key assumption is the L^2 boundedness of the operator. When the operator is of convolution type, $Tf = f * k$ for some $k \in \mathcal{S}'$, boundedness of T in $L^2(\mathbb{R}^d)$ is equivalent to $m = \hat{k}$ being bounded, and this is how we have checked this property till now in most cases.

For operators of non-convolution type, boundedness in $L^2(\mathbb{R}^d)$ may not be an easy thing. There is however a general result, due originally to Cotlar, that covers some interesting applications.

<u>Cotlar's lemma</u>. Let H be a Hilbert space and $\{T_j\}_{j\in\mathbb{Z}}$ a family of bounded linear operators $T_j : H \to H$.

We assume that for some ~~fixed~~ sequence $a : \mathbb{Z} \to \mathbb{R}^+$

$$\|T_j^* T_k\| + \|T_j T_k^*\| \leq a(j-k)$$

with $$A = \sum_{j\in\mathbb{Z}} \sqrt{a(j)} < +\infty. \text{ Then}$$

(a) For all finite subsets $\Lambda \subset \mathbb{Z}$, $\|\sum_{j\in\Lambda} T_j\| \leq A$

(b) For all $x \in H$, $\sum_{j\in\mathbb{Z}} \|T_j(x)\|^2 \leq A^2 \|x\|^2$

(c) For all $x \in H$, $\sum_{|j|\leq N} T_j(x)$ has limit $:= T(x)$

and the operator T thus defined is bounded with $\|T\| \leq A$.

To understand Cotlar's lemma, let us consider the case $a(j) = \delta(j)$. Then the assumption means that the ranges of the T_j are mutually orthogonal, and hence $\sum T_j$ obviously makes sense. In general, since $a(j) \to 0$ as $|j| \to +\infty$, the assumption means that $R(T_j)$ and $R(T_k)$ become more and more orthogonal as $|j-k|$ growths. <u>(almost orthogonality)</u>

Notice that for $j=k$ the ~~lemma~~ hypothesis includes

$$\|T_j\|^2 \leq a(0) \leq A^2$$

Proof of Cotlar's lemma. We use the elementary fact that for any operator S, one has

$$\|S\|^2 = \|SS^*\|$$

In particular, if S is self-adjoint, $\|S\|^2 = \|S^2\|$, and iterating, we have that $\|S\|^m = \|S^m\|$ if S is self-adjoint and m is a power of 2. Applying this to SS^*

$$\|SS^*\|^m = \|(SS^*)^m\|$$

we get

$$\|S\|^2 = \|(SS^*)^m\|^{1/m}$$

We apply this to $S = \sum_{j \in \Lambda} T_j$; for this

$$(SS^*)^m = \sum_{j_1, j_2, \dots, j_{2m} \in \Lambda} T_{j_1} T_{j_2}^* T_{j_3} T_{j_4}^* \dots T_{j_{2m-1}} T_{j_{2m}}^*$$

$$\|(SS^*)^m\| \le \sum_{j_1, j_2, \dots, j_{2m} \in \Lambda} \|T_{j_1} T_{j_2}^* \dots T_{j_{2m}}^*\|$$

We estimate in two different ways: first we group together $(j_1, j_2), (j_3, j_4), \dots, (j_{2m-1}, j_{2m})$ and use the hypothesis to bound by

$$a(j_1 - j_2)\, a(j_3 - j_4) \cdots a(j_{2m-1} - j_{2m})$$

Next we leave j_1, j_{2m} alone and group $(j_2, j_3), (j_4, j_5)$ etc to bound by

$$\sqrt{a(0)}\, a(j_2-j_3)\, a(j_4-j_5) \cdots a(j_{2m-2}-j_{2m-1})\sqrt{a(0)}$$

we now use the geometric mean of both estimates, that is,

$$\| T_{j_1} T_{j_2}^* T_{j_3} T_{j_4}^* \cdots T_{j_{2m-1}} T_{j_{2m}}^* \| \le$$

$$\le \sqrt{a(0)}\sqrt{a(j_1-j_2)}\sqrt{a(j_2-j_3)} \cdots \sqrt{a(j_{2m-1}-j_{2m})}$$

Summing first in j_1, then in j_2, etc we get

$$\|(SS^*)^m\| \le \sqrt{a(0)}\; A^{2m-1}\; \#(\Lambda)$$

and hence

$$\|\sum_{j\in\Lambda} T_j\|^2 \le a(0)^{1/2m}\, A^{\frac{2m-1}{m}}\, |\Lambda|^{1/m}$$

and letting $m\to+\infty$ we get (a).

To prove (b) we need the Rademacher functions r_j. These are functions in $[0,1]$, $|r_j|=1$ such that

$$\int_0^1 r_j(s)\, r_k(s)\, ds = 0 \qquad j\neq k.$$

This implies that for fixed x

$$\int_0^1 \Big\| \sum_{j\in\Lambda} r_j(s)\, T_j(x) \Big\|^2 ds =$$

$$= \int_0^1 \Big(\sum_{j\in\Lambda} \|T_j(x)\|^2 + \sum_{\substack{j,k\in\Lambda \\ j\neq k}} r_j(s)\, r_k(s) < T_j(x), T_k(x) > \Big) ds$$

$$= \sum_{j\in\Lambda} \|T_j(x)\|^2 .$$

Fixed $s\in[0,1)$ we apply the first part to the operators $r_j(s)\,T_j$, that satisfy the same hypotheses, because $|r_j| = 1$. That is, $\| \sum_{j\in\Lambda} r_j(s)\, T_j \| \leq A$; using the above relation we obtain part (b).

For part (c) we must see that for given $x\in H$ the sequence
$$\sum_{j=-N}^{+N} T_j(x)$$
is Cauchy. Suppose not, then exists $\varepsilon > 0$ and $N_k \nearrow +\infty$ such that
$$\Big\| \sum_{N_k \leq |j| < N_{k+1}} T_j(x) \Big\| \geq \varepsilon .$$

Fixed $s\in[0,1)$, we now apply part (a) to
$$\tilde{T}_j = r_k(s)\, T_j \qquad \text{if} \quad N_k \leq |j| < N_{k+1}$$

(that still satisfy the same hypothesis), and obtain

$$\left\| \sum_{k=1}^{K} r_k(s) \sum_{N_k \le |j| < N_{k+1}} T_j(x) \right\| \le A \|x\|$$

If we square and integrate in s we arrive at

$$\sum_{k=1}^{K} \left\| \sum_{N_k \le |j| < N_{k+1}} T_j(x) \right\|^2 \le A^2 \|x\|^2$$

which is a contradiction. //

We will see now two applications of Cotlar's lemma. First we will show the L^2-boundedness of the Hilbert transform (without using Fourier transform) and secondly we will ~~see the~~ consider pseudodifferential operators of order zero.

Example. We will prove the L^2-boundedness of

$$Hf(x) = \text{p.v.} \frac{1}{\pi} \int \frac{f(x-t)}{t}\, dt$$

Set

$$T_j f(x) = \int_{2^j < |t| < 2^{j+1}} \frac{f(x-t)}{t}\, dt$$

Each T_j is obviously bounded in $L^2(\mathbb{R})$ and

$$|T_j f(x)| \leq 4\, Mf(x)$$

where Mf denotes the Hardy-Littlewood max. fcn.

To apply Cotlar's lemma we need estimating $T_i T_j^*$.
But $T_j^* = -T_j$ so we estimate $\|T_i T_j\|$.

We have $T_j f = K_j * f$ with $K_j(t) = \frac{1}{t} \mathbb{1}_{\Delta_j}(t)$
$\Delta_j = \{2^j < |t| < 2^{j+1}\}$. Then

$$T_i T_j f = K_i * K_j * f$$

$$\|T_i T_j\| \leq \|K_i * K_j\|_1 .$$

So let us compute $K_i * K_j$

$$K_i * K_j(x) = \int \frac{1}{t} \mathbb{1}_{\Delta_i}(t) \frac{1}{x-t} \mathbb{1}_{\Delta_j}(x-t)\, dt$$

Assume $i<j$, $x>0$, without loss of generality

First notice that $K_i * K_j(x) = 0$ if $x \notin (2^j - 2^{i+1}, 2^{j+1} + 2^{i+1})$

Also

$$|K_i * K_j(x)| \leq \int_{\Delta_i} \frac{1}{|t|} 2^{-j} dt \leq 2 \cdot 2^{-j}$$

This estimate is enough for the integral of $|K_i * K_j(x)|$ in the two intervals

$$(2^j - 2^{i+1}, 2^j + 2^{i+1}), \quad (2^{j+1} - 2^{i+1}, 2^{j+1} + 2^{i+1})$$

that have both length $4\,2^i$, that is, the integral of $|K_i * K_j|$ over them is $\leq 8\,2^{i-j}$

For the remaining interval $(2^j + 2^{i+1}, 2^{j+1} - 2^{i+1})$ that has length $\sim 2^{-j}$ we need a better estimate of $K_i * K_j(x)$ using cancellation. Indeed, if x is in this interval, and $t \in \Delta_i$ then $x - t \in \Delta_j$, i.e

$$K_i * K_j(x) = \int \frac{1}{t} \mathbb{1}_{\Delta_i}(t) \frac{1}{x-t} dt =$$

$$= \int \frac{1}{t} \mathbb{1}_{\Delta_i}(t) \left\{ \frac{1}{x-t} - \frac{1}{x} \right\} dt = \int \mathbb{1}_{\Delta_i}(t) \frac{dt}{x(x-t)}$$

$$\to \quad |K_i * K_j(x)| \leq 2 \cdot 2^i\, 2^{-2j} \Rightarrow \|K_i * K_j\|_1 \leq C\, 2^{|i-j|}$$

and we are done.

Pseudodifferential operators

Let us start repeating the (formal) manipulations we did in the section "Singular integrals with variable kernel". For a differential operator with variable coefficients we may write

$$\sum_\alpha c_\alpha(x) D^\alpha f(x) = \int_{\mathbb{R}^d} a(x,\xi)\, e^{2\pi i x\cdot\xi} \hat{f}(\xi)\, d\xi, \quad f \in S$$

with

$$a(x,\xi) = \sum_\alpha c_\alpha(x)\,(2\pi i\xi)^\alpha$$

Motivated by this, for any smooth symbol

$$a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$$

of slow growth in ξ, we define the operator

$$a(X,D) f(x) = \int_{\mathbb{R}^d} a(x,\xi)\, \hat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi, \quad f \in S(\mathbb{R}^d)$$

Formally, if we apply $a(X,D)$ to $f = e^{2\pi i x\cdot\eta}$ for which $\hat{f} = \delta_\eta$ we get $a(X,D) f(x) = a(x,\eta) f(x)$, so the symbol can be recovered from $a(X,D)$.

These operators generalize both the multipliers (for which $a(x,\xi) = m(\xi)$ only depends on ξ) and the multiplication operators $M(x) f(x)$, for which $a(x,\xi)$ only depends on x. Note that in general

$$(ab)(X,D) \neq a(X,D)\, b(X,D)$$

<u>Definition</u> (Standard pseudodifferential operators). A smooth (C^∞) symbol a is called <u>standard of order $k \in \mathbb{R}$</u> if

$$|D_x^\gamma D_\xi^\beta a(x,\xi)| = O\left(1+|\xi|\right)^{k-|\beta|}$$

and write $a \in S^k$. The corresponding $a(X,D)$

$$a(X,D) f(x) = \int_{\mathbb{R}^d} a(x,\xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \quad , \quad f \in S(\mathbb{R}^d)$$

is well-defined on $f \in S(\mathbb{R}^d)$ (because $|a(x,\xi)| = O(1+|\xi|)^k$) and is called a <u>standard pseudo-differential operator</u> (ψDO)

In fact

$$a(X,D) : S(\mathbb{R}^d) \longrightarrow S(\mathbb{R}^d)$$

To see this we imitate the usual proof that the Fourier transform maps $S(\mathbb{R}^d)$ to itself. We consider the operator

$$L_\xi = \frac{I - \Delta_\xi}{(1 + 4\pi^2 |x|^2)}$$

that satisfies $L_\xi^N e^{2\pi i x \cdot \xi} = e^{2\pi i x \cdot \xi}$. Integrating by parts N-times

$$a(X,D) f(x) = \int e^{2\pi i x \cdot \xi} (L_\xi)^N \left[a(x,\xi) \hat{f}(\xi) \right] d\xi .$$

$$= \frac{1}{(1+4\pi^2 |x|^2)^N} \int e^{2\pi i x \cdot \xi} (I - \Delta_\xi)^N \left[a(x,\xi) \hat{f}(\xi) \right] d\xi$$

$$= O\left(\frac{1}{(1+|x|)^{2N}} \right)$$

The same argument applies to the derivatives D_x^α of $a(X,D)f(x)$ whose symbol is $D_x^\alpha a$.

Leibnitz rule implies $ab \in S^{k+\ell}$ if $a \in S^k, b \in S^\ell$. Also, $D_x^\alpha D_\xi^\beta a \in S^{k-|\beta|}$ if $a \in S^k$.

At this point we would like to obtain the kernel representation of $a(X,D)$. First we operate formally replacing $\hat{f}(\xi)$ by its expression and interchanging the order of integration

$$a(X,D)f(x) = \int_{\mathbb{R}^d} \underbrace{\left\{ \int_{\mathbb{R}^d} a(x,\xi)\, e^{2\pi i (x-y)\cdot\xi}\, d\xi \right\}}_{k(x,\,x-y) = K(x,y)} f(y)\, dy$$

The problem with this is that the ξ-integral is not absolutely convergent. To avoid this problem one approximates the symbol $a(x,\xi)$ by symbols of compact support as follows:

choose $\gamma \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, $\gamma = 1$ near the origin and set

$$a_\varepsilon(x,\varepsilon) = a(x,\xi)\, \gamma(\varepsilon x, \varepsilon \xi)$$

Then $a_\varepsilon \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, $a_\varepsilon \in S^k$ uniformly in ε, that is

$$|D_x^\alpha D_\xi^\beta a_\varepsilon(x,\xi)| \le C\,(1+|\xi|)^{k-|\beta|} \qquad \forall \varepsilon,\ \varepsilon < 1$$

Also, $D_x^\alpha D_\xi^\beta a_\varepsilon \to D_x^\alpha D_\xi^\beta a$ as $\varepsilon \to 0$. This implies that the corresponding ΨDO $a_\varepsilon(X,D)$ satisfy

$$a_\varepsilon(X,D) f(x) \to a(X,D) f(x) \qquad f \in S(\mathbb{R}^d).$$

The practical consequence of this is that when proving estimates on $a(X,D)$ we may assume that a has compact support, if the estimates just depend on the S^k constants and not on the support of a.

Assuming a compactly supported we will now obtain an estimate of $K(x,y) = k(x, x-y)$, a first one

$$k(x,z) = \int_{\mathbb{R}^d} a(x,\xi)\, e^{2\pi i z\cdot\xi}\, d\xi$$

For fixed x, $k(x,\cdot)$ is thus the inverse Fourier transform of $a(x,\cdot)$. Then $(-2\pi i z)^\alpha D_z^\beta k(x,z)$ is the inverse Fourier transform of $D_\xi^\alpha\left[(2\pi i\xi)^\beta a(x,\xi)\right]$. Since $a \in S^k$ one has

$$\left| D_\xi^\alpha\left[(2\pi i\xi)^\beta a(x,\xi)\right]\right| = O\left(1+|\xi|\right)^{k+|\beta|-|\alpha|}$$

This will be in $L^1(\mathbb{R}^d)$ with respect to ξ if $|\alpha| > k+|\beta|+d$;
Therefore

$$(-2\pi i z)^\alpha D_z^\beta k(x,z)$$

will be bounded if $|\alpha| > k+|\beta|+d$. If we replace k by $D_x^\gamma k$ ($a(x,\xi)$ by $D_x^\gamma a(x,\xi)$) the same holds

$$(-2\pi i z)^\alpha D_z^\beta D_x^\gamma k(x,z)$$

will be bounded if $|\alpha| > k+|\beta|+d$

<u>THEOREM</u> (Calderon-Vaillancourt). Every pseudodifferential operator of order 0 is a CZO, and in particular is bounded in $L^p(\mathbb{R}^d)$, $1<p<\infty$.

<u>Proof</u>. We need proving <u>two</u> things

(a) The Hörmander type conditions,

$$|K(x,y)| = O(|x-y|^{-d})$$

$$|\nabla_x K(x,y)| = O(|x-y|^{-d-1})$$

$$|\nabla_y K(x,y)| = O(|x-y|^{-d-1})$$

(b) The $L^2(\mathbb{R}^d)$-boundedness

For part (a), notice first that the considerations before the statement (for $|\beta|=0$) give ~~[illegible]~~

$$|K(x,y)| = O(|x-y|^{-m}) \qquad m>d$$

but now we will do better.

(d) The Hörmander type condition. It is convenient to write $a_x(\xi) = a(x,\xi)$, so that $k_x(z) = k(x,z) = (a_x)^\vee(z)$. Then

$$(-2\pi i z)^\alpha k(x,z) = (-2\pi i z)^\alpha k_x(z)$$

is the inverse Fourier transform of $D_\xi^\alpha[a_x]$.

We saw the following lemma when dealing with CZO of convolution type.

<u>Lemma</u>. Assume that $h \in C_c^\infty(\mathbb{R}^d)$ satisfies

$$|h(\xi)| \le \frac{A}{|\xi|^d}, \quad |\nabla h(\xi)| \le \frac{A}{|\xi|^{d+1}}$$

$$\left| \int_{R_1<|\xi|<R_2} h(\xi)\,d\xi \right| \le A$$

Then $\|h^\vee\|_\infty \le C(d)\, A$

We apply this to $h(\xi) = D_\xi^\alpha a_x(\xi)$. Then, since $a \in S^0$

$$|h(\xi)| = |D^\alpha a_x(\xi)| = |D^\alpha a(x,\xi)| \le A(1+|\xi|)^{-|\alpha|}$$

$$|\nabla h(\xi)| \le A(1+|\xi|)^{-|\alpha|-1}$$

Also, if we choose $|\alpha| = d$.

$$\left| \int_{R_1<|\xi|<R_2} h(\xi)\,d\xi \right| \leq \int_{R_1<|\xi|<R_2} \frac{d\xi}{(1+|\xi|)^d} \leq A$$

where the constant A only depends on the S^0-norm of a.

Applying the lemma we obtain

$$|k(x,z)| \leq \frac{C(d)A}{|z|^d}$$

Similarly $\nabla_y K(x,y) = \nabla_y k(x,x-y)$ $(-2\pi i z)^\alpha \nabla_z k(x,z)$ is the inverse Fourier transform of $D^\alpha_\xi \left[(2\pi i \xi)\, a(x,\xi) \right]$, and we apply the same argument for $h(\xi) = D^\alpha_\xi \left[(2\pi i \xi)\, a(x,\xi) \right]$

Now

$$|h(\xi)| \leq A\,(1+|\xi|)^{1-|\alpha|}$$

$$|\nabla h(\xi)| \leq A\,(1+|\xi|)^{-|\alpha|}$$

and to get

$$\left| \int_{R_1<|\xi|<R_2} h(\xi)\,d\xi \right| \leq A$$

we need now $|\alpha| = d+1$. Thus we get

$$|\nabla_z k(x,z)| = O\left(|z|^{-d-1} \right)$$

$$|\nabla_y K(x,y)| = O\left(|x-y|^{-d-1} \right)$$

with constants depending only on the S^0 constants of a.

We finally note that

$$(\nabla_x + \nabla_y) K(x,y) = \int (\nabla_x a)(x,\xi)\, e^{2\pi i (x-y)\cdot\xi}\, d\xi$$

Since $\nabla_x a$ is also in S^0, this has ~~we have by~~ the same estimate ~~as~~ K, that is

$$|(\nabla_x + \nabla_y) K(x,y)| = O\left(|x-y|^{-d} \right)$$

and therefore

$$|\nabla_y K(x,y)| = O\left(|x-y|^{-d-1} \right).$$

(b) Boundedness in $L^2(\mathbb{R}^d)$ is of course equivalent to boundedness in $L^2(\mathbb{R}^d)$ of

$$T_a(f)(x) = \int_{\mathbb{R}^d} a(x,\xi)\, f(\xi)\, e^{2\pi i x\xi}\, d\xi$$

(by Plancherel's theorem). ~~We~~ We will break T_a into pieces and apply Cotlar's Lemma.

Fix $\varphi(\xi)$ supported in $[-\varepsilon, 1+\varepsilon]^d$, $\varphi \geq 0$, C^∞ and such that

$$\sum_{j\in\mathbb{Z}^d} \varphi(x-j) = 1, \quad x \in \mathbb{R}^d$$

For $j,k \in \mathbb{Z}^d$ we define symbols

$$a_{jk}(x,\xi) = \varphi(x-j)\, a(x,\xi)\, \varphi(\xi-k)$$

and the corresponding operators $T_{j,k}$ as above with a replaced by $a_{j,k}$. Then

$$T_a = \sum_{j,k} T_{j,k}$$

(T_{jk} is the piece corresponding to $x \sim j$, $\xi \sim k$).

We will prove that

$$\|T_{jk} T^*_{j'k'}\|, \|T^*_{jk} T_{j'k'}\| \leq C_N \frac{1}{(1+|j-j'|+|k-k'|)^{2N}}$$

uniformly in j,k,j',k'. This will imply boundedness in

$L^2(\mathbb{R}^d)$ because

$$\sum_{j,k\in\mathbb{Z}^d} \sqrt{\frac{1}{(1+|j|+|k|)^{2N}}} \leq$$

$$\leq \sum_{j\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|j|)^{N/2}} \frac{1}{(1+|k|)^{N/2}} < +\infty$$

if N is big enough (e.g. $N \geq 2d+2$).

The kernel of $T_{j,k}^* T_{j',k'}$ is

$$K_{j,k,j',k'}(x,y) = \int_{\mathbb{R}^d} \overline{a_{j,k}(z,x)}\, a_{j',k'}(z,y)\, e^{2\pi i (y-x)\cdot z}\, dz$$

$$T_{j,k}^* T_{j',k'} f(x) = \int_{\mathbb{R}^d} K_{j,k,j',k'}(x,y) f(y)\, dy$$

We integrate by parts using

$$e^{2\pi i (y-x)\cdot z} = \frac{(I-\Delta_z)^N \left(e^{2\pi i (y-x)\cdot z} \right)}{(1+4\pi^2|x-y|^2)^N}.$$

$$K_{j,k,j',k'}(x,y) = \cancel{\int_{\mathbb{R}^d}} \frac{1}{(1+4\pi^2|x-y|^2)^N} \int e^{2\pi i (y-x)\cdot z} (I-\Delta_z)^N \left[\overline{a_{j,k}(z,x)}\, a_{j',k'}(z,y) \right]$$

$$= \pm \frac{\varphi(x-k)\,\varphi(y-k')}{(1+4\pi^2|x-y|^2)^N} \int e^{2\pi i (y-x)\cdot z} (I-\Delta_z)^N \left[\varphi(z-j)\,\overline{\sigma(z,x)}\,\sigma(z,y)\,\varphi(z-j') \right] dz$$

This is not zero only if $|j-j'| \leq 2\sqrt{d}$, and also $1+|x-y| \sim 1+|k-k'|$. Hence

$$|K_{j,k,j',k'}(x,y)| \leq \begin{cases} C_N \dfrac{\varphi(x-k)\,\varphi(y-k')}{(1+|k-k'|)^{2N}}, & |j-j'| \leq 2\sqrt{d} \\ 0 \quad \text{if} & |j-j'| > 2\sqrt{d} \end{cases}$$

This implies

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K_{j,k,j',k'}(x,y)|\,dy \leq \frac{C_N}{(1+|j-j'|+|k-k'|)^{2N}}$$

$$\sup_{y} \int |K_{j,k,j',k'}(x,y)|\,d\cancel{y}x \leq \text{same}$$

By Schur's lemma, $T_{j,k}^* T_{j',k'}$ ~~has~~ is bounded in $L^2(\mathbb{R}^d)$ with norm

$$\|T_{j,k}^* T_{j',k'}\| \leq \frac{C_N}{(1+|j-j'|+|k-k'|)^{2N}}.$$

For $T_{j,k} T_{j',k'}^*$ is exactly the same proof.