Lecture notes, Singular Integrals

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Chapter 2

Interpolation theory

In this chapter we will review the main tools in interpolation theory used in Harmonic Analysis, namely the Riesz-Thorin convexity theorem, and the Marcinkiewicz theorem. The later will be presented in the context of Lorentz spaces, that will appear later as well in the context of regularity for Sobolev spaces.

2.1 The Riesz-Thorin interpolation theorem

We are assuming that T is a linear operator acting on a domain $D \subset L^0(X) :\to L^0(Y)$, and we assume that D contains the simple functions and is closed by truncation. Recall that we say that T is of strong-type $(p, q), 1 \leq p, q \leq +\infty$, with constant C if

$$\|Tf\|_{L^q(Y)} \leq C\|f\|_{L^p(X)}, f \in D.$$

Theorem 1. Assume that T is of strong-type $(p_i, q_i), i = 0, 1, 1 \leq p_i, q_i \leq +\infty$ with norm $k_i, i = 0, 1$. For $0 < t < 1$ define p_t, q_t by inverse convex combination

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Then T is of type (p_t, q_t) with constant less than or equal to $k_0^{1-t} k_1^t$. That is, $\log \|T\|_t$ is convex in t.

We begin with the so-called three lines lemma, a strip version of the well-known three circles-lemma.

Lemma 1. Let Π denote the vertical strip $\Pi = \{z = t + is : 0 < t < 1\}$. Let F be a bounded holomorphic function in Π, continuous on $\overline{\Pi}$. Assume that

$$|F(is)| \leq M_0, |F(1 + is)| \leq M_1.$$

Then $|F(t + is)| \leq M_0^{1-t} M_1^t$.

Proof. We may assume that $M_0, M_1 > 0$. Then we consider the holomorphic function

$$G(z) = M_0^{1-z} M_1^z = e(1-z)\log M_0 + z \log M_1.$$

Notice that this function is up to multiplicative constants the only bounded non vanishing holomorphic function on Π, continuous on $\overline{\Pi}$ having modulus M_0 on $i\mathbf{R}$ and modulus M_1 in $1 + i\mathbf{R}$. This amounts to saying that $(1-t)\log M_0 + t \log M_1$ is the only bounded harmonic function in Π and continuous on $\overline{\Pi}$ which equals $\log M_0$ on $i\mathbf{R}$ and $\log M_1$ on $1 + i\mathbf{R}$.

By replacing F by F/G we may then assume that $M_0 = M_1 = 1$, and want to see that $|F(z)| \leq 1$. If we knew that $\lim_{|s| \to +\infty} |F(t+is)| = 0$ uniformly in t, then this is a consequence of the maximum modulus principle. In general we consider $F_n(z) = F(z)e^{(z^2-1)/n}$ and notice that

$$|F_n(z)| = |F(z)| e^{\frac{1}{n}(t^2 - s^2 - 1)} \leq |F(z)| e^{-\frac{s^2}{n}},$$

so that $|F_n| \leq 1$ on the boundary and $\lim_{|y| \to +\infty} |F_n(t+is)| = 0$ uniformly in t. Hence $|F_n(z)| \leq 1$ and making $n \to \infty$ we conclude. $\square$

Proof. To prove the theorem, we consider

$$\alpha(z) = (1-z)\frac{1}{p_0} + z\frac{1}{p_1}, \beta(z) = (1-z)\frac{1}{q_0} + z\frac{1}{q_1},$$

and fix $t, p = p_t, q = q_t$. It is enough to prove that for f, g simple functions

$$f = \sum_j a_j 1_{E_j}, g = \sum_k b_k 1_{F_k},$$

and $\|f\|_p = \|g\|_{q'} = 1$, one has

$$|I| = |\int_Y (Tf) g d\nu| \leq k_0^{1-t} k_1^t.$$

Assume first that $p < +\infty, q > 1$; we consider now functions f_z, g_z defined as follows: if $f(x) = |f(x)|\omega, g(x) = |g(x)|\eta, |\omega| = |\eta| = 1$, set

$$f_z(x) = |f(x)|^{p\alpha(z)}\omega, g_z(x) = |g(x)|^{q'(1-\beta(z))}.$$

Notice that $f_t = f, g_t = g$ and that

$$|f_{is}| = |f|^{p\Re((1-is)\frac{1}{p_0} + is\frac{1}{p_1})} = |f|^{\frac{p}{p_0}}, |g_{is}| = |g|^{\frac{q'}{q'_0}},$$

and similarly

$$|f_{1+is}| = |f|^{\frac{p}{p_1}}, |g_{1+is}| = |g|^{\frac{q'}{q'_1}}.$$

Then,

$$I(z) = \int_Y (Tf_z) g_z d\nu,$$

satisfies the hypothesis of the lemma with

$$|I(is)| \leq k_0 \|f_{is}\|_{p_o} \|g_{is}\|_{q_0'} \leq k_0 (\int_X |f|^p d\mu)^{\frac{1}{p_0}} (\int_Y |g(y)|^{q'} d\nu)^{\frac{1}{q_0'}} = k_0,$$

and similarly $|I(1+is)| \leq k_1$. Hence $|I| = |I(t)|$ is bounded by $k_0^{1-t} k_1^t$ as claimed.

In case $p = \infty, q = 1$ then $p_0 = p_1 = \infty, q_0 = q_1 = 1$ and there is nothing to prove. If $p < +\infty$ and $q = 1$ the same argument works with $g_z = g$, while if $p = \infty, q > 1$ take $f_z = f$. □

An application is Young's inequality for integral operators $Tf(y) = \int_X K(x,y) f(x) d\mu(x)$ given by a kernel K satisfying

$$\int_X |K(x,y)|^r d\mu(x) \leq C^r, \int_Y |K(x,y)|^r d\nu(y) \leq C^r.$$

The first one implies that T is of strong type (r', ∞) simply by Holder's inequality, while the second one implies that T is of strong type $(1, r)$ using the continuous Minkowski's inequality. It then follows T is of strong type (p,q) with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$.

Another application concerns the Fourier transform, which is of strong types $(1, \infty)$ (by just a question of size) and $(2,2)$ (by a matter of cancellation). In this situation, the Fourier transform is also defined for $f \in L^p(\mathbf{R^d}), \mathbf{1} < \mathbf{p} < \mathbf{2}$ by writing $f = f_1 + f_2, f_1 \in \mathrm{L}^1(\mathbf{R^d}), \mathbf{f_2} \in \mathbf{L^2}(\mathbf{R^d})$ (simply define $f_1 = f$ if $|f_1| > 1$ and zero elsewhere) and setting $\hat{f} = \hat{f}_1 + \hat{f}_2$, a definition that is obviously independent of the chosen decomposition. This definition agrees with the definition of $\hat{f}$ as a tempered distribution.

The Riesz-Thorin theorem implies the Haussdorf-Young theorem:

Theorem 2. $\|\hat{f}\|_{p'} \leq \|f\|_p, 1 \leq p \leq 2$

Notice that the definition is in terms of L^p-convergence:

$$\hat{f}(\xi) = \lim_{r \to +\infty} \int_{|x| \leq r} f(x) e^{2\pi i \xi \cdot x} dx.$$

For $1 < p < 2$, it can be proved the a.e. existence of this limit in reasonable terms but for $p = 2$ is a very deep and hard result by L. Carleson and Hunt.

Also notice that if $f \in L^p(\mathbf{R^d}), \mathbf{g} \in \mathbf{L^r}(\mathbf{R^d}), \mathbf{1} \leq \mathbf{p}, \mathbf{r} \leq \mathbf{2}$ and $\frac{1}{p} + \frac{1}{r} \geq \frac{3}{2}$, then $f * g \in \mathrm{L}^q(\mathbf{R^d})$ with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1, 1 \leq q \leq 2$, by Young's inequality, then $\widehat{f * g}$ makes sense, is in $L^{q'}(\mathbf{R^d})$, with $\frac{1}{q'} = \frac{1}{p'} + \frac{1}{q'}$, and equals $\hat{f}\hat{g}$.

One might ask if there are other pairs (p,q) such that the Fourier transform is of strong type (p,q), say

$$\|\hat{\varphi}\|_q \leq C \|\varphi\|_p, \varphi \in \mathcal{S}(\mathbf{R^d}).$$

Applying this to the dilate $\varphi_\lambda(x) = \lambda^{-d}\varphi(\frac{x}{\lambda})$, for which $\widehat{\varphi_\lambda}(\xi) = \widehat{\varphi}(\lambda\xi)$, we get that

$$\lambda^{-\frac{d}{q}}\|\widehat{\varphi}\|_q \leq C\lambda^{\frac{d}{p}-d}\|\varphi\|_p.$$

Making $\lambda \to 0$ and $\lambda \to \infty$ we see that necessarily $q = p'$.

The Haussdorf-Young theorem says that this holds if $1 \leq p \leq 2$. Now we will see that it does not hold if $p > 2$. Indeed, let us consider the gaussian $g(x) = e^{-\pi|x|^2}$ for which $\hat{g} = g$ and consider a linear combination of the type

$$\varphi(x) = \sum_{n=1}^{N} e^{2\pi i x\cdot v_n} g(x - u_n).$$

This function is called a translate of g in the time-frequency plane. Introducing the notation $M_x f(\xi) = e^{2\pi i x\cdot\xi} f(\xi)$ for the multiplication operator by e_x, recall that $\widehat{\tau_a f} = M_{-a}\hat{f}$ and conversely, $\widehat{M_b f} = \tau_b(\hat{f})$, that is $\widehat{M_b\tau_a f} = \tau_b M_a \hat{f}$. This means that

$$\hat{\varphi}(\xi) = \sum_{n=1}^{N} e^{-2\pi i x\cdot u_n} g(x - v_n),$$

so choosing $u_n = v_n$ we have that $\hat{\varphi} = \overline{\varphi}$. Now, as $|v_n| \to +\infty$ the N-terms of φ behave like if they had disjoint supports, and $\|\varphi\|_p$ is about $N^{\frac{1}{p}}$. So, if there is a strong type (p, q) inequality we would have $N^{\frac{1}{q}} \leq CN^{\frac{1}{p}}$ for all N, whence $p' = q \geq p$.

2.2 Lorentz spaces

If we have a discrete function, that is, a finite number of values $a_1, \cdots, a_N$, it is clear the meaning of a "reordering" of these numbers. We would like to define reordering for general functions in $\mathbf{R^d}$, with the purpose of defining new functions spaces, other than the L^p- spaces, that capture with more precision properties of functions.

We are in a general measure space (X, μ), although the main example to have in mind is $\mathbf{R^d}$. If f is measurable we defined the distribution function $\lambda_f(s)$ by $\lambda_f(s) = \mu\{|f| > s\}$, a decreasing function in $\mathbf{R}^+$, so that

$$\int_X \varphi(|f|)d\mu = \int_{\mathbf{R}^+} \varphi'(s)\lambda_f(s)\, ds,$$

for all increasing φ. We now define the non-increasing rearrangement f^* of f by

$$f^*(t) = \inf\{s : \lambda_f(s) \leq t\}.$$

It is easily seen that both are right-continuous. By definition, $f^*(t) > s$ means $\lambda_f(s) > t$ so that $f^{**} = f^*$, that is, f and f^* have the same distribution function and so the same L^p-integrals, etc.

Suppose that f is a simple function $f(x) = \sum_j a_j 1_{E_j}(x)$, with the E_j pairwise disjoint and $a_1 > a_2 > \cdots < a_N$. Then $\lambda_f(s) = 0$ for $s \geq a_1$, equals $t_1 = \mu(E_1)$ for $a_2 \leq s < a_1$, equals $t_2 = \mu(E_1) + \mu(E_2)$ for $a_3 \leq s < a_2$ and so on, equal to $t_N = \sum_{j=1}^N \mu(E_j)$ for $0 \leq s < a_N$, so it is a step function with jumps $\mu(E_j)$ at a_j. Then f^* takes the value a_1 in $[0, t_1), a_2$ in $[t_1, t_2)$, and so on, that is, it takes the same values of f on sets with the same measure, in decreasing size. Of course, if we identify a discrete function $a_1, \cdots, a_N$ with the function $\sum a_j 1_{(j-1,j)}$, then the rearrangement of the function gives back the same numbers rearranged in decreasing order.

The Lorentz spaces $L^{p,r}(X)$ are defined for $r < +\infty$ for all $p, 1 \leq p < +\infty$ by

$$\|f\|_{p,r} = \left(\frac{r}{p} \int_0^\infty [t^{\frac{1}{p}} f^*(t)]^r \frac{dt}{t} \right)^{\frac{1}{r}} < +\infty,$$

and for $r = \infty, 1 \leq p \leq +\infty$ by using the weak L^p-norms

$$\|f\|_{p,\infty} = \sup t^{\frac{1}{p}} f^*(t) < +\infty.$$

The spaces $L^{\infty,r}$ with r finite are not defined as the condition is satisfied only by $f = 0$. This definition is motivated by the fact that

$$\|f\|_{p,p} = \|f^*\|_p = \|f\|_p, 1 \leq p \leq +\infty.$$

Also notice that for a characteristic function 1_E,

$$\|1_E\|_{p,r} = \left(\frac{r}{p} \int_0^{\mu(E)} t^{\frac{r}{p}-1} dt \right)^{\frac{1}{r}} = \mu(E)^{\frac{1}{p}},$$

does not depend on r.

The following properties hold:

- $L^{p,r}(X)$ is a linear space. Since what is involved is the L^r-norm with respect to $\frac{dt}{t}$ of $(t^{\frac{1}{p}} f^*(t)$, it is enough to show that

  $$(f+g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2),$$

  and use it for $t_1 = t_2 = \frac{t}{2}$ to find that $\|f+g\|_{p,r} \leq C_{p,r}(\|f\|_{p,r} + \|g\|_{p,r}$. In turn, the above follows from

  $$\lambda_{f+g}(s_1 + s_2) \leq \lambda_f(s_1) + \lambda_g(s_2),$$

  which is obvious. Although the $\|f\|_{p,r}$ are not true norms, it is possible to find equivalent true norms if $p > 1$, but we will not need them.

- The "norm" $\|f\|_{p,r}$ can be defined in terms solely of λ_f, more concretely in terms of $\Phi(s) = s\lambda_f(s)^{\frac{1}{p}}$ as follows. For $r = \infty$ it is clear that by the very definition

$$\sup_t t^{\frac{1}{p}} f^*(t) = \sup_s s\lambda_f(s)^{\frac{1}{p}} = \sup_s \Phi(s).$$

For $r < \infty$ (strictly speaking the following argument is correct just in case λ is strictly decreasing but helps understand the basic idea) we make the change of variables $s = f^*(t), t = \lambda_f(s)$, to get

$$\|f\|_{p,r} = \left(-\frac{r}{p}\int_0^{+\infty} \lambda_f(s)^{\frac{r}{p}-1} s^r d(\lambda_f(s))\right)^{\frac{1}{r}} = \left(-\int_0^{+\infty} s^r d(\lambda_f^{\frac{r}{p}})\right)^{\frac{1}{r}}$$
$$= \left(r\int_0^{+\infty} \lambda_f(s)^{\frac{r}{p}} s^{r-1} ds\right)^{\frac{1}{r}} = \left(r\int_0^{+\infty} \Phi(s)^r \frac{ds}{s}\right)^{\frac{1}{r}}.$$

- The norms $\|f\|_{p,r}$ decrease in r for fixed p, $\|f\|_{p,r_2} \leq \|f\|_{p,r_1}$ if $r_1 < r_2$ so that $L^{p,r_1}(X) \subset L^{p,r_2}(X)$. We will prove here this inequality up to a constant, and it just depends on f^* being decreasing. First, for $r_2 = \infty$, for all t,

$$\|f\|_{p,r_1} \geq \left(\frac{r_1}{p}\int_0^t [u^{\frac{1}{p}} f^*(u)]^{r_1} \frac{du}{u}\right)^{\frac{1}{r_1}} \geq f^*(t)\left(\frac{r_1}{p}\int_0^t u^{\frac{r_1}{p}-1} du\right)^{\frac{1}{r_1}} = t^{\frac{1}{p}} f^*(t),$$

and the result follows taking supremum in t. If r_2 is finite, we break the inner power r_2 in $r_2 - r_1$ (that we bound by the sup $\|f\|_{p,\infty}$) and r_1 to obtain

$$\|f\|_{p,r_2} \leq \|f\|_{p,\infty}^{\frac{r_2-r_1}{r_2}} \left(\frac{r_2}{p}\int_0^{\infty} [t^{\frac{1}{p}} f^*(t)]^{r_1} \frac{dt}{t}\right)^{\frac{1}{r_2}},$$

and use the already proved to bound it by

$$\|f\|_{p,r_1}^{\frac{r_2-r_1}{r_2}} C(r_2, r_1)\|f\|_{p,r_1}^{\frac{r_2}{r_1}} = C(r_2, r_1)\|f\|_{p,r_1}.$$

- Defining convergence in terms of $\|f\|_{p,r}$, the Lorentz space is complete, simple functions are dense if $r < +\infty$ and the convergence always implies convergence in measure.

Proposition 1. For f, g measurable functions on X one has

$$\int_X |fg| d\mu \leq \int_0^{+\infty} f^*(t) g^*(t)\, dt.$$

As a consequence the Hlder inequality holds true

$$\int_X |fg| d\mu \le \|f\|_{p,r} \|g\|_{p',r'},$$

(when p or p' are infinite then r or r' respectively, must be infinite too).

Proof. One has

$$\int_X |fg| d\mu = \int_X \int_{0<u<|f(x)|} \int_{0<v<|g(x)|} dudvd\mu \le$$

$$\int_0^{+\infty} \int_0^{+\infty} \mu(\{x : |f(x)| > u, |g(x)| > v\}) dudv$$

$$\le \int_0^{+\infty} \int_0^{+\infty} \min(\lambda_f(u), \lambda_f(v)) dudv =$$

$$\int_0^{+\infty} \int_0^{+\infty} \min(|\{f^*(t) > u\}|, |\{g^*(t) > v\}|) dudv$$

$$= \int_0^{+\infty} \int_0^{+\infty} |\{f^*(t) > u\} \cap \{g^*(t) > v\}| dudv = \int_0^{+\infty} f^*(t) g^*(t) dt.$$

From this,

$$\int_X |fg| d\mu \le \int_0^{+\infty} t^{\frac{1}{p}} f^*(t) t^{\frac{1}{p'}} g^*(t) \frac{dt}{t}$$

and Hlder's inequality follows from the usual Hlder's inequality with respect the measure $\frac{dt}{t}$. □

Hlder's inequality implies that

$$\int_E |f| d\mu \le \|f\|_{p,r} \|1_E\|_{p',r'} = \|f\|_{p,r} \mu(E)^{\frac{1}{p'}},$$

valid if $p > 1$ for all r and also for $p = r = 1$, so that in this cases functions of $L^{p,r}(X)$ are integrable over sets of finite measure.

It can be proved too for $p > 1$ than the converse Holder inequality holds as well, that is, $\|f\|_{p,r}$ is comparable to

$$\sup\{|\int_X fg d\mu| : \|g\|_{p',r'} \le 1\},$$

the right term defining then a true norm equivalent to $\|f\|_{p,r}$.

The following is due to Riesz:

Proposition 2. Let Mf denote the Euclidian (uncentered) maximal function of f, both in $\mathbf{R^d}$ and $\mathbf{R}$. Then

$$(Mf)^*(t) \le CM(f^*)(t) = C\frac{1}{t} \int_0^t f^*(u) du.$$

Proof. By what was proved in the previous chapter we know that

$$s\lambda_{Mf}(Cs) \leq \int_{|f|>s} |f(x)|d\mu(x).$$

Now note that

$$\int_{|f|>s} |f(x)|d\mu(x) = s\lambda_f(s) + \int_s^\infty \lambda_f(t)\,dt = \int_0^{\lambda_f(s)} f^*(u)du,$$

and so taking $s = f^*(t)$ we get that

$$\int_{|f|>f^*(t)} |f(x)|d\mu(x) = \int_0^t f^*(u)du = tM(f^*)(t).$$

Write $\rho = M(f^*)(t)$. Combining both and noticing that $\rho \geq f^*(t)$, we see that

$$\rho\lambda_{Mf}(C\rho) \leq \int_{|f|>\rho} |f|d\mu \leq \int_{|f|>f^*(t)} |f|d\mu = t\rho,$$

that is, $\lambda_{Mf}(C\rho) \leq t$, which means that $(Mf)^*(t) \leq \rho = M(f^*)(t)$. □

In dimension $d = 1$ one can show using the Riesz rising sun lemma that the above holds with $C = 1$, The Riesz inequality implies that for every increasing φ one has

$$\int_X \varphi((Mf)(x))d\mu(x) \leq C \int_X \varphi(M(f^*))d\mu.$$

This was the original inequality obtained by Hardy-Littlewood; in $d = 1$ and for discrete functions it says that if $a_1, a_2, \cdots, a_N$ are arbitrary numbers and the $b_1 \geq b_2 \geq \cdots \geq b_N$ denote the same numbers in decreasing order, and if

$$\alpha_k = \max_{1\leq j\leq k} \frac{\sum_{i=j}^k a_i}{k-j+1}, \beta_k = \frac{1}{k}\sum_{i=1}^k b_i,$$

then for every increasing φ it holds

$$\sum_{k=1}^N \varphi(\alpha_k) \leq \sum_{k=1}^N \varphi(\beta_k).$$

Hardy, a very good player of cricket, said that the best way to understand it is thinking that the a_i represent the cricket scores and φ the batsman "total satisfaction", so the theorem says that "the batsman's total satisfaction is maximized if he plays a given collection of innings in decreasing order".

For $\varphi(t) = t^p$, and using Hardy's inequality one can obtain another proof of the strong (p, p) inequality for Mf for $p > 1$.

2.3 The Marcinkiewicz interpolation theorem

The Riesz-Thorin interpolation theorem uses complex methods, deals with strong type conditions and applies to linear operators. Next we will describe the Marcinkiewicz interpolation theorem that applies to subadditive maps and Lorentz spaces. In fact, Lorentz spaces are defined so to extract the maximum information from the hypothesis. When applied to classical Lebesgue spaces $L^p(\mathbf{R^d})$ the Marcinkiewicz theorem does not imply the Riesz-Thorin theorem in full generality.

Theorem 3. 1. Assume that T is a subadditive operator acting on a domain $D \subset L^0(X) :\to L^0(Y)$, and assume that D contains the simple functions and is closed by truncation. Assume too that with $p_0 < p_1, q_0 \neq q_1, 1 \leq p_i, q_i \leq +\infty$,

$$\|Tf\|_{q_i,\infty} \leq A_i \|f\|_{p_i,1}, f \in D.$$

(Note that this is the case under the stronger assumption that T is of weak type $(p_i, q_i), i = 0, 1$). Then, if

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}, 0 < t < 1,$$

T satisfies

$$\|Tf\|_{q_t,r} \leq B(t, A_1, A_2) \|f\|_{p_t,r}, f \in D, 1 \leq r \leq \infty$$

In fact it is enough that the hypothesis holds for $f = 1_E$, a characteristic function.

2. In particular, one has

$$\|Tf\|_{q_t,p_t} \leq C\|f\|_{p_t}, \|Tf\|_{q_t} \leq C\|f\|_{p_t,q_t}.$$

In case $p_1 \leq q_1$, then $p_t \leq q_t$ and

$$\|Tf\|_{q_t} \leq C\|f\|_{p_t}.$$

Proof. We will give the proof of a weakened version and only in the particular case that $p_i = q_i$, that's what we will need later. The proof in the general case can be found in [SteinWeiss, Introduction to Fourier Analysis on Euclidean spaces, pg. 197]. We assume that T is of weak type (p_0, p_0) and $(p_1, p_1), p_0 < p_1$,

$$\|Tf\|_{p_i,\infty} \leq A_i \|f\|_{p_i}, f \in D, i = 0, 1,$$

and will prove that T is of strong type (p, p) for $p_0 < p < p_1$. For $f \in D \cap L^p(\mathbf{R^d})$ and $s > 0$ given we decompose $f = f_0 + f_1$, with $f_0 = f$

where $|f| > cs$, and zero elsewhere, with the constant c to be chosen. Then $|Tf| \leq |Tf_0| + |Tf_1|$ and hence

$$\lambda_{Tf}(s) \leq \lambda_{Tf_0}(\frac{s}{2}) + \lambda_{Tf_1}(\frac{s}{2}).$$

If $p_1 = \infty$, we have $\|Tf_1\|_\infty \leq A_1\|f_1\|_\infty \leq A_1 cs$, so choosing $c = 1/(2A_1)$ the second term is zero, while the first is bounded by

$$\left(\frac{2A_0}{s}\|f_0\|_{p_0}\right)^{p_0}.$$

Now, using Fubini

$$\|Tf\|_p^p = p\int_0^\infty s^{p-1}\lambda_{Tf}(s)ds \leq 2A_0)^p p\int_0^\infty s^{p-1-p_0}(\int_{|f|>cs}|f(x)|^{p_0}d\mu)ds =$$

$$= C\int_X |f(x)|^{p_0}\int_0^{|f(x)|/c} s^{p-1-p_0}ds = C\|f\|_p^p.$$

Assume now that p_1 is finite; then

$$\lambda_{Tf_i}(\frac{s}{2}) \leq \left(\frac{2A_i\|f_i\|_{p_i}}{s}\right)^{p_i}, i = 0, 1,$$

and therefore

$$\|Tf\|_p^p = p\int_0^\infty s^{p-1}\lambda_{Tf}(s)ds \leq$$

$$(2A_0)^p p\int_0^\infty s^{p-1-p_0}(\int_{|f|>cs}|f(x)|^{p_0}d\mu)ds + (2A_1)^p p\int_0^\infty s^{p-1-p_1}(\int_{|f|<cs}|f(x)|^{p_1}d\mu)ds =$$

$$= \left(\frac{p2^{p_0}}{p-p_0}\frac{A_0^{p_0}}{c^{p-p_0}} + \frac{p2^{p_1}}{p_1-p}\frac{A_1^{p_1}}{c^{p-p_1}}\right)\|f\|_p^p.$$

By minimizing in c in the bound just obtained one may check that the constant B obtained satisfies

$$B \leq 2p^{\frac{1}{p}}\left(\frac{1}{p-p_0} + \frac{1}{p_1-p}\right)^{\frac{1}{p}} A_0^{1-t}A_1^t.$$

□

2.4 Applications of the Marcinkiewicz theorem

As a first application the Haussdorf-Young theorem can be improved. Since the Fourier transform is of strong types $(1, \infty)$ and $(2, 2)$ we get

Theorem 4. For $1 < p < 2, 1 \leq r \leq \infty$, $\|\hat{f}\|_{p',r} \leq C_p\|f\|_{p,r}$. In particular, $\|\hat{f}\|_{p',p} \leq C\|f\|_p$, $\|\hat{f}\|_{p'} \leq C\|f\|_{p,p'}$.

The second application will show how Marcinkiewicz result can be used to improve inequalities in an automatic way. Let us consider again the operator of convolution with a fixed $g \in L^r$, that as we saw satisfies strong type $(1, r)$ and (r', ∞). By Riesz theorem we saw that

$$\|f * g\|_q \leq \|f\|_p \|g\|_r,$$

for $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$, that is, for $1 \leq p \leq r'$ and q given by that equation. But with Marcinkiewicz theorem we can improve and find that for $1 < p < r'$, the same q and all $s, 1 \leq s \leq \infty$, also we have

$$\|f * g\|_{q,s} \leq C\|f\|_{p,s}.\|g\|_r.$$

This already says that to get $f * g \in L^q$ we need just $f \in L^{p,q}$, a larger space than L^p. But in fact more can be said if instead we think now with f fixed in $L^{p,\infty}(X), 1 < p$, as an operator in g. The last estimate with $s = \infty$ tells us that it is of weak type (r, q) for $1 < r < p'$ and q given by the same equation. But this is an open range of weak inequalities; given a fixed r in this range we can always choose r_0, r_1 so that $r_0 < r < r_1$, the corresponding q_0, q and q_1 will satisfy the same convex relationships than the $r's$ and therefore we can conclude by another application of Marcinkiewicz theorem that in fact a strong inequality holds. Altogether we have proved the *Young's inequality for weak type spaces*

$$\|f * g\|_q \leq \|f\|_{p,\infty} \|g\|_r, 1 < p, q, r < \infty.$$

Finally, the Hardy-Littlewood maximal operator Mf satisfies a weak $(1, 1)$ inequality and a strong (∞, ∞) inequality, and so an application of Marcinkiewicz theorem shows that M is bounded in all $L^{p,q}(\mathbf{R^d}), \mathbf{1 < p}$.

The theorems of Riesz-Thorin and Marcinkiewicz constitute a glympse to what is calles *Interpolation theory* in Functional Analysis. The general context is the same, that is, one has an operator that satisfies two different estimates, say it is bounded from E_0 to F_0 and from E_1 to F_1, with E_0, E_1 lying in a common ambiance space and similarly for F_0, F_1. One then *constructs* the intermediate spaces E_t, F_t so that T is bounded too from E_t to F_t. Two different approacles exist, the real method, based on K-functional, and the complex method that uses complex analysis in the spirit of the proof of the Riesz-Thorin theorem.

By both of these theories, the interpolated space between L^{p_0}, L^{p_1} is L^{p_t}, and we have thus avoided the general construction.