

Lecture notes, Singular Integrals

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Chapter 3

The Hilbert transform

In this chapter we will study the Hilbert transform. This is a specially important operator for several reasons:

- Because of its relationship with summability for Fourier integrals in L^p -norms.
- Because it constitutes a link between real and complex analysis
- Because it is a *model case* for the general theory of singular integral operators.

A main keyword in the theory of singular integrals and in analysis in general is *cancellation*. We begin with some easy examples of what this means.

3.1 Some objects that exist due to cancellation

One main example of something that exists due to cancellation is the Fourier transform of functions in $L^2(\mathbb{R}^d)$. In fact the whole L^2 -theory of the Fourier transform exists thanks to cancellation properties. Let us review for example the well-known Parseval's theorem, stating that for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ one has $\|\hat{f}\|_2 = \|f\|_2$, a result that allows to extend the definition of the Fourier transform to $L^2(\mathbb{R}^d)$. Formally,

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) \overline{f(y)} \left(\int_{\mathbb{R}^d} e^{2\pi i \xi \cdot (y-x)} d\xi \right) dx dy.$$

This being equal to $\int_{\mathbb{R}^d} |f(x)|^2 dx$ means formally that

$$\int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} d\xi = \delta_0(x).$$

Along the same lines, let us look at the Fourier inversion theorem, stating that whenever $f \in L^1(\mathbb{R}^d)$, $\hat{f} \in L^1(\mathbb{R}^d)$ one has

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi.$$

Again, the right hand side, by a formal use of Fubini's theorem becomes

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(y) e^{-2\pi i \xi \cdot y} dy \right) e^{2\pi i \xi \cdot x} d\xi = \int_{\mathbb{R}^d} f(y) \left(\int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot (y-x)} d\xi \right) dy.$$

If this is to be equal to $f(x)$ we arrive to the same formal conclusion, namely that superposition of all frequencies is zero outside zero. An intuitive way to understand this is by noting that ($d = 1$)

$$\int_{-R}^R e^{2\pi i \xi x} d\xi = \frac{\sin(2\pi R x)}{\pi x},$$

is zero for x an integer multiple of $\frac{1}{R}$, so when $R \rightarrow +\infty$ the zeros become more and more dense.

The Fourier transform exists trivially if $f \in L^1(\mathbb{R}^d)$, but not so trivially if $f \in L^2(\mathbb{R}^d)$. It exists, in the L^2 -sense

$$\hat{f}(\xi) = \lim_n \int_{\mathbb{R}} f_n(x) e^{-2\pi i x \xi} dx,$$

with f_n arbitrary in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that $f_n \rightarrow f$ in $L^2(\mathbb{R}^d)$. The most trivial choice is to take $f_R(x)$ to be equal to $f(x)$ if $|x| < R$ and zero elsewhere, so that

$$\hat{f}(\xi) = \lim_R \int_{-R}^R f(x) e^{-2\pi i x \xi} dx,$$

the limit being in the L^2 -sense. It is a general and very deep result due to L. Carleson that this limits exists a.e.

To illustrate all this we will consider the Fourier transform of two particular functions, none of which in $L^1(\mathbb{R})$, and study the pointwise behaviour of the limit above.

For instance, $\int_{\mathbb{R}} \frac{\sin x}{x} dx$ is conditionally convergent, meaning that

$$\lim_{R \rightarrow \infty} \int_{-R}^R \frac{\sin x}{x} dx \tag{3.1}$$

exists while $\int_{\mathbb{R}} \left| \frac{\sin x}{x} \right| dx = +\infty$. The later can be proved by estimating from below the contribution to the integral at points $|x - (\frac{\pi}{2} + k\pi)| < \delta, k \in \mathbb{Z}$. Alternatively, noting that the Fourier transform of $1_{[-a,a]}$ is

$$\int_a^a e^{-2\pi x \xi} dx = \frac{\sin 2\pi a \xi}{\pi \xi},$$

we can argue that it cannot be integrable, because if it were, by Fourier inversion theorem, $1_{[-a,a]}$ would be continuous. We note too that by Parseval's theorem

$$\int_{\mathbb{R}} \left(\frac{\sin 2\pi a\xi}{\pi\xi} \right)^2 d\xi = \int_{\mathbb{R}} 1_{[-a,a]}^2 = 2a.$$

Regarding the value of the integral, we will see that the limit

$$\lim_{R \rightarrow +\infty} \int_{-R}^R \frac{\sin 2\pi a\xi}{\pi\xi} e^{2\pi i\xi x} d\xi = 1_{[-a,a]}(x), \quad (3.2)$$

that as we know exists in the L^2 sense, exists too pointwise a.e., in fact for $x \neq \pm a$. Indeed, by using residues we have that

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{-R}^R \frac{e^{i\alpha\xi}}{\xi} d\xi$$

equals $\frac{1}{2} \text{Res}\left(\frac{e^{i\alpha\xi}}{\xi}, 0\right) = \frac{1}{2}$ if $\alpha > 0$ and $-\frac{1}{2}$ if $\alpha < 0$, and this implies (??). For $x = 0$ we get that

$$\lim_{R \rightarrow +\infty} \int_{-R}^R \frac{\sin 2\pi a\xi}{\pi\xi} d\xi = 1,$$

from which it follows that the integral in (??) equals π .

Another example are the Fresnel integrals

$$\int_{\mathbb{R}} \sin x^2 dx, \int_{\mathbb{R}} \cos x^2 dx.$$

By the change of variable $u = x^2$ one has

$$\int_0^R \sin x^2 dx = \frac{1}{2} \int_0^{R^2} \frac{\sin u}{\sqrt{u}} du,$$

so the integral is not absolutely convergent. To compute the integral we will see more generally that

$$\lim_{R \rightarrow +\infty} \int_{-R}^R e^{-\alpha\pi x^2} e^{-2\pi i x\xi} dx = 2 \lim_{R \rightarrow +\infty} \int_0^R e^{-\alpha\pi x^2} \cos 2\pi x\xi dx = \frac{1}{\sqrt{\alpha}} e^{-\frac{\pi}{\alpha}\xi^2}. \quad (3.3)$$

for $\text{Re } \alpha \geq 0$ and all $\xi \in \mathbb{C}$. For $\text{Re } \alpha > 0$, the above is the Fourier transform of $f_\alpha(x) = e^{-\alpha\pi x^2}$, a function in the Schwarz class. Since it is holomorphic in α and for α, ξ real equals $\frac{1}{\sqrt{\alpha}} e^{-\frac{\pi}{\alpha}\xi^2}$, the same holds true by analytic continuation first in α and then in ξ , that is

$$F(\alpha, \xi) = \int_{\mathbb{R}} e^{-\alpha\pi x^2} e^{-2\pi i x\xi} dx = \frac{1}{\sqrt{\alpha}} e^{-\frac{\pi}{\alpha}\xi^2}, \text{Re } \alpha > 0, \xi \in \mathbb{C}.$$

By making $\operatorname{Re} \alpha \rightarrow 0$ we can already conclude that the Fourier transform, in the sense of distributions, of $e^{-\alpha\pi x^2}$ is $\frac{1}{\sqrt{\alpha}}e^{-\frac{\pi}{\alpha}\xi^2}$ also if $\operatorname{Re} \alpha = 0$. However, this does not imply (??). To compute the limit in (??) when $\operatorname{Re} \alpha = 0$ we will exploit the analyticity in x as follows. We consider the contour in the z -plane given by the segment $[0, R]$, the arc $t \mapsto Re^{i\pi t}$, $0 \leq t \leq \frac{1}{4}$, and the segment from $Re^{i\frac{\pi}{4}}$ to 0, and apply the Cauchy theorem to the entire function $e^{-\alpha\pi z^2} \cos(2\pi z\xi)$. The contribution of the arc is bounded by aRe^{-bR^2+cR} with some positive constants a, b, c depending on ξ , and so has limit zero when $R \rightarrow \infty$. Hence, parametrizing with $z = t\frac{1+i}{\sqrt{2}}$, we get

$$\lim_{R \rightarrow +\infty} \int_0^R e^{-\alpha\pi x^2} \cos(2\pi x\xi) dx = \frac{1+i}{\sqrt{2}} \lim_{R \rightarrow \infty} \int_0^R e^{-\alpha\pi it^2} \cos(2\pi \frac{1+i}{\sqrt{2}} t\xi) dt.$$

If $\operatorname{Re} i\alpha > 0$, that is, $\operatorname{Im} \alpha < 0$ this shows that the limits exist and equals

$$\frac{1+i}{\sqrt{2}} F(i\alpha, \frac{1+i}{\sqrt{2}} \xi) = \frac{1}{\sqrt{\alpha}} e^{-\frac{\pi}{\alpha}\xi^2},$$

as claimed. For $\xi = 0$ and $\alpha = -\frac{i}{\pi}$ we get the value $\frac{\sqrt{\pi}}{2\sqrt{2}}$ for the Fresnel integrals.

3.2 The conjugate Poisson kernel and the Cauchy transform

One method to study the summability of the Fourier integral is by the means method. The Abel method consists in using

$$\int_{\mathbb{R}^d} \hat{f}(\xi) e^{-2\pi t|\xi|} e^{2\pi i x \cdot \xi} d\xi.$$

The behaviour as $t \rightarrow 0$ is easily understood. Indeed, by Fubini's theorem the above equals

$$\int_{\mathbb{R}^d} f(y) \left(\int_{\mathbb{R}^d} e^{-2\pi t|\xi|} e^{2\pi i \xi(x-y)} d\xi \right) dy = (f * P_t)(x),$$

with

$$P_t(x) = \int_{\mathbb{R}^d} e^{-2\pi t|\xi|} e^{2\pi i x \cdot \xi} d\xi.$$

Let us compute explicitly P_t in $d = 1$:

$$P_t(x) = \int_0^{+\infty} e^{2\pi \xi(ix-t)} d\xi + \int_{-\infty}^0 e^{2\pi \xi(ix+t)} d\xi = A + B,$$

with

$$A = \frac{1}{2\pi(t-ix)}, B = \bar{A} = \frac{1}{2\pi(t+ix)}.$$

This gives

$$P_t(x) = \frac{1}{\pi} \frac{t}{x^2 + t^2}.$$

Note that $P_t(x) = \frac{1}{t}P(x/t)$, with $P = P_1$, and that $P_1, e^{-2\pi|\xi|}$ are a Fourier pair, P is positive with integral one. In dimension $d > 1$ the computation is harder, the result being $P_t(x) = t^{-d}P(x/t)$, with

$$P(x) = c_d \frac{1}{(|x|^2 + 1)^{\frac{d+1}{2}}}, c_d = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}},$$

It is convenient to look as $(f * P_t)(x)$ as a function of (x, t) in the upper half space $t > 0$. Noticing that P is in all L^p -spaces we define

Definition 1. For $f \in L^p(\mathbb{R}^d), 1 \leq p \leq +\infty$, we define the *Poisson transform* $u = P[f]$ as the function defined in the upper half-space $t > 0$ by

$$P[f](x, t) = (f * P_t)(x) = \frac{1}{\pi} \int_{\mathbb{R}^d} f(y) \frac{t}{(x-y)^2 + t^2} dy = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{-2\pi t|\xi|} e^{2\pi i x \cdot \xi} d\xi$$

Since we will need it again in a more general form we state again the main properties of convolutions with functions of the form $h_t(x) = t^{-d}h(x/t)$, with $h \in L^1(\mathbb{R}^d)$.

Proposition 1. Let $h \in L^1(\mathbb{R}^d), m = \int h$, and set $h_t(x) = t^{-d}h(x/t)$. For $f \in L^p(\mathbb{R}^d), 1 \leq p \leq +\infty$, consider $f * h_t$. Then

- $\|f * h_t\|_p \leq \|f\|_p \|h\|_1$, and $f * h_t \rightarrow mf$ as $t \rightarrow 0$ in $L^p(\mathbb{R}^d), 1 \leq p < +\infty$.
- If f is bounded and continuous at x_0 then $f * h_t(x_0) \rightarrow mf(x_0)$.
- If f is bounded and uniformly continuous then $f * h_t \rightarrow mf$ uniformly.
- If the *least radially decreasing majorant* of h ,

$$\Psi(x) = \sup_{|y| \geq |x|} |h(y)|,$$

is in $L^1(\mathbb{R}^d)$, then $f * h_t(x) \rightarrow mf(x)$ a.e.

Proof. The continuous Minkowski inequality implies the first assertion. We write

$$f * h_t(x) = \int_{\mathbb{R}^d} f(x-y) h_t(y) dy, f * h_t(x) - mf(x) = \int_{\mathbb{R}^d} (f(x-y) - f(x)) h_t(y) dy,$$

so that

$$|f * h_t(x) - mf(x)| \leq \int_{\mathbb{R}^d} |f(x-ty) - f(x)| |h(y)| dy,$$

and by continuous Minkowski inequality again

$$\|f * h_t - mf\|_p \leq \int_{\mathbb{R}^d} \|\tau_{ty}f - f\|_p |h(y)| dy.$$

The dominated convergence theorem implies then all assertions but the last. We saw in the first chapter that

$$\sup_t |f * h_t(x)| \leq \|\Psi\|_1 Mf(x),$$

so that the maximal function on the left satisfies a weak (1, 1) estimate, hence it suffices to check the last assertion for f continuous with compact support, and this is the second statement. \square

Theorem 1. Let $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq +\infty$ and let $u = P[f]$ be its Poisson transform. Then

- The function u is harmonic in the upper half-space $t > 0$.
- The function u satisfies

$$\sup_t \left(\int_{\mathbb{R}^d} |u(x, t)|^p dt \right)^{\frac{1}{p}} \leq \|f\|_p, 1 \leq p < \infty, |u(x, t)| \leq \|f\|_\infty, p = +\infty. \quad (3.4)$$

- $\lim_{t \rightarrow 0} u(x, t) = f(x)$ a.e. In fact the non-tangential limit

$$\lim_{(y, t) \rightarrow x, |y-x| \leq ct} u(y, t)$$

is a.e. equal to $f(x)$.

The fact that u is harmonic is checked on the Fourier transform side definition. The other properties are restatements of proposition ?? in the particular case of the Poisson kernel. The last statement is checked first for f continuous with compact support and then it follows from the fact that the non-tangential maximal function is also bounded by Mf for general f .

The space of harmonic functions in the half-space satisfying (??) is denoted by $h^p(\mathbb{R}_+^{d+1})$. It is possible, for $1 < p \leq +\infty$ and using the Helly selection theorem (that is Banach-Alaouglu theorem for L^p - spaces) to prove that it is exactly the space of harmonic functions that are Poisson transforms of functions in $L^p(\mathbb{R}^d)$. For $p = 1$ it is the space of Poisson transforms $P[\mu](x, t) = (\mu * P_t)(x)$ of finite complex Borel measures μ in \mathbb{R}^d . If $u \in h^1$, and moreover, $\lim_t u(\cdot, t) = f$ exists in L^1 , then $u = P[f]$.

We now consider only $d = 1$. We know that an harmonic function u in a simply connected domain has up to constants a unique harmonic conjugate v , that is, such that $u + iv$ is holomorphic. Let us compute explicitly the

harmonic conjugate of $u = P[f]$. For this purpose we rewrite, in terms of $z = x + it$,

$$u(z) = P[f](z) = \int_0^{+\infty} \hat{f}(\xi) e^{2\pi i z \xi} d\xi + \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z} \xi} d\xi$$

which exhibits $P[f]$ as the sum of an holomorphic function and a antiholomorphic function.

We define $v(z)$ by setting

$$iv(z) = \int_0^{+\infty} \hat{f}(\xi) e^{2\pi i z \xi} d\xi - \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z} \xi} d\xi,$$

so that v is also harmonic and

$$(u + iv)(z) = 2 \int_0^{+\infty} \hat{f}(\xi) e^{2\pi i z \xi} d\xi,$$

is indeed holomorphic.

This is the expression of v in the Fourier transform side. On the other side we can use the same computations done above. Indeed, to define iv we have separated the contribution of positive and negative ξ and changed the sign of the later. This corresponds to considering, in the above notations, $A - B = A - \bar{A}$ instead of $A + B$. Since

$$A - \bar{A} = \frac{1}{2\pi} \left(\frac{1}{t - ix} - \frac{1}{t + ix} \right) = \frac{i}{\pi} \frac{x}{x^2 + t^2},$$

this means that $v(x, t) = (f * Q_t)(x)$, where

$$Q_t(x) = \frac{1}{\pi} \frac{x}{x^2 + t^2}, \quad \widehat{Q}_t(\xi) = -i \operatorname{sign}(\xi) e^{-2\pi t |\xi|}.$$

Note that $Q_t(x) = \frac{1}{t} Q(x/t)$, with $Q(x) = \frac{1}{\pi} \frac{x}{x^2 + 1}$ and that

$$P_t(x) + iQ_t(x) = \frac{1}{\pi} \frac{t + ix}{x^2 + t^2} = \frac{i}{\pi z}.$$

The family Q_t is called the *conjugate Poisson kernel*. Notice that $Q \notin L^1(\mathbb{R})$ but $Q \in L^q(\mathbb{R})$ for $1 < q \leq +\infty$.

Definition 2. If $f \in L^p(\mathbb{R})$, $1 \leq p < +\infty$ we define the *conjugate Poisson transform* $Q[f]$ of f as the harmonic function in the upper half-plane defined by

$$Q[f](z) = (f * Q_t)(x) = \frac{1}{\pi} \int_{\mathbb{R}} f(x - y) \frac{y}{y^2 + t^2} dy, \quad z = x + it, \quad t > 0,$$

and the *Cauchy transform* by

$$Cf(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(y)}{y - z} dy, \quad z \notin \mathbb{R}.$$

The computations above show that

$$Cf(z) = \frac{1}{2}(P[f] + iQ[f])(z) = \int_0^{+\infty} \hat{f}(\xi)e^{2\pi iz\xi} d\xi, t = \text{Im } z > 0.$$

They also show that

$$(P[f] - iQ[f])(z) = 2 \int_{-\infty}^0 \hat{f}(\xi)e^{2\pi i\bar{z}\xi} d\xi = -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(y)}{y - \bar{z}} dy,$$

or

$$Cf(z) = \frac{1}{2}(-P[f](\bar{z}) + iQ[f](\bar{z})), \text{Im } z < 0.$$

3.3 The Hilbert transform

We want to study now the behaviour of $f * Q_t$ as $t \rightarrow 0$. Formally we have

$$\lim_{t \rightarrow 0} Q_t(x) = \frac{1}{\pi} \frac{1}{x}, \lim_{t \rightarrow 0} \widehat{Q}_t(\xi) = -i \text{sign}(\xi).$$

Convolution with $\frac{1}{x}$ does not mean sense since it is not integrable and neither is a distribution. Now, recall that thanks to Parseval's theorem, any bounded measurable function m in \mathbb{R} defines a translation invariant operator T in $L^2(\mathbb{R})$ by setting $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$, and then $\|T\| = \|m\|_{\infty}$. In our case, then, it makes sense

Definition 3. The Hilbert transform H is the translation-invariant operator defined in $L^2(\mathbb{R})$ by

$$\widehat{Hf}(\xi) = -i \text{sign}(\xi)\hat{f}(\xi),$$

or, equivalently,

$$Hf = \lim_{t \rightarrow 0} f * Q_t, \text{ in } L^2(\mathbb{R}).$$

Theorem 2. The Hilbert transform H satisfies:

- It is an isometry of $L^2(\mathbb{R})$ with $H^* = H^{-1} = -H$, hence

$$\int Hfg = - \int fHg, f, g \in L^2(\mathbb{R}).$$

- With the notation $\rho_{\lambda}f(x) = f(\lambda x)$, $\lambda > 0$ for positive dilations and $\tilde{f}(x) = f(-x)$ for the reflection operator, H commutes with the ρ_{λ} , $H\rho_{\lambda} = \rho_{\lambda}H$, and anticommutes with the reflection operator, $H(\tilde{f})(x) = -\widehat{Hf}(x) = -Hf(-x)$.
- If T is a bounded operator in $L^2(\mathbb{R})$ that commutes with translations, with positive dilations, and anticommutes with the reflection operator, then $T = cH$ for some constant c .

Proof. Let us call $m(\xi) = -i \operatorname{sign}(\xi)$ for the multiplier of H . The first statement then corresponds to $|m| = 1, \bar{m} = m^{-1} = -m$. Now, one has $\widehat{\rho_\lambda f}(\xi) = \hat{f}(\xi/\lambda)$, and $\widehat{\tilde{f}}(\xi) = \hat{f}(-\xi) = \tilde{\hat{f}}(\xi)$; therefore, a translation invariant operator with multiplier m commutes with positive dilations iff m is a constant on \mathbb{R}^+ , and it anticommutes with the reflection iff m is odd. Thus, H has these properties and any other operator with the same properties is a scalar multiple of H . \square

Notice that the same argument shows that the multiples of the identity operator are the only ones that commute with translations, with positive dilations and with the reflection.

Thus we have a clear picture of what is $\lim_t Q_t$ on the Fourier transform side and now we would like to understand it as a convolution operator. Formally,

$$\lim_t Q_t(x) = \frac{1}{\pi} \frac{1}{x},$$

which is not a locally integrable function, neither it is a distribution, it does not make sense something like

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{y-x} dy.$$

At this point we recall the definition of the so called its *principal value*, namely,

$$\lim_{t \rightarrow 0} \frac{1}{\pi} \int_{|y-x|>t} \frac{f(y)}{x-y} dy = \frac{1}{\pi} \int_{|y|>t} \frac{f(x-y)}{y} dy.$$

We will set $\Phi_t(x) = \frac{1}{x} \mathbf{1}_{|x|>t}$, and

$$H_t f(x) = \frac{1}{\pi} (f * \Phi_t)(x) = \frac{1}{\pi} \int_{|y|>t} \frac{f(x-y)}{y} dy.$$

The functions Φ_t , that also obey $\Phi_t(x) = \frac{1}{t} \Phi(x/t)$, with $\Phi = \Phi_1$, have limit in the distribution sense, that is, the limit

$$\lim_{t \rightarrow 0} \langle \Phi_t, \varphi \rangle = \lim_{t \rightarrow 0} \int_{|y|>t} \frac{\varphi(y)}{y} dy,$$

exists for every $\varphi \in \mathcal{S}(\mathbb{R}^\Gamma)$. This is because exploiting that the kernel is odd the limit equals

$$\int_0^{+\infty} \frac{\varphi(y) - \varphi(-y)}{y} dy = \int_{|y|<1} \frac{\varphi(y) - \varphi(0)}{y} dy + \int_{|y|>1} \frac{\varphi(y)}{y} dy.$$

The limit of Φ_t in the distribution sense is called $p.v. \frac{1}{x}$. Thus,

$$\lim_{t \rightarrow 0} H_t \varphi(x) = \frac{1}{\pi} \varphi * (p.v. \frac{1}{x}) = \lim_{t \rightarrow 0} \frac{1}{\pi} \int_{|y| > t} \frac{\varphi(x-y)}{y} dy = \int_0^\infty \frac{\varphi(x-y) - \varphi(x+y)}{y} dy,$$

exists for all $\varphi \in \mathcal{S}(\mathbb{R}^1)$ and all x . Now we will see that in fact it exists for general $\varphi \in L^2$, and equals $H(\varphi)$.

Theorem 3. For $f \in L^2(\mathbb{R})$, the Hilbert transform, that has been defined as

$$Hf = \lim_{t \rightarrow 0} f * Q_t, \text{ in } L^2(\mathbb{R}),$$

or equivalently, by $\widehat{Hf}(\xi) = -i \operatorname{sign}(\xi) \hat{f}(\xi)$, is also given by

$$Hf = \frac{1}{\pi} (f * p.v. \frac{1}{x}) = \lim_{t \rightarrow 0} H_t f = \lim_{t \rightarrow 0} \frac{1}{\pi} f * \Phi_t,$$

that is

$$Hf(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(x-y)}{y} dy,$$

where this limit is in L^2 -norm as well. Moreover, for both expressions

$$\lim_{t \rightarrow 0} f * Q_t(x) = \lim_{t \rightarrow 0} H_t f(x) = Hf(x) \text{ a.e.}$$

Proof. First note that for $f \in L^2(\mathbb{R})$ we have that $Q_t f = P_t Hf$ because both have the same Fourier transform $-i \operatorname{sign}(\xi) e^{-2\pi t |\xi|} \hat{f}(\xi)$. Since P_t is an approximation of the identity with a radial function, it follows that $Q_t f(x) = P_t(Hf)(x)$ has limit $Hf(x)$ a.e. Next we look at the difference $h_t(x) = Q_t(x) - \frac{1}{\pi} \Phi_t(x)$, that also obeys the rule $h_t(x) = \frac{1}{t} h(x/t)$, with

$$h(x) = \frac{1}{\pi} \begin{cases} \frac{x}{1+x^2}, & |x| < 1 \\ \frac{x}{1+x^2} - \frac{1}{x} = -\frac{1}{x(1+x^2)}, & |x| > 1 \end{cases}.$$

Since h is odd, it has integral $m = \int h = 0$; moreover, its radial majorant Ψ is explicitly, up to the factor $\frac{1}{\pi}$, equal to $\frac{1}{|x|(1+|x|^2)}$ if $|x| > 1$ and so it is integrable. Applying the proposition ??, we see that $\lim_{t \rightarrow 0} f * h_t = 0$, for $f \in L^p(\mathbb{R})$, $1 \leq p < +\infty$, both in L^p -norm and pointwise a.e. Since for $f \in L^2(\mathbb{R})$, the limit of $f * Q_t$ equals Hf both in L^2 -norm and pointwise a.e., the same happens with $H_t f$. \square

That completes the definition of H in $L^2(\mathbb{R})$, from three points of view: through a multiplier $-i \operatorname{sign}(\xi)$ of $L^2(\mathbb{R})$, through a complex analysis point of view (limit of the conjugate harmonic function $Q[f]$), and from the real analysis point of view (as a principal value integral), the limits existing both in the L^2 -sense and a.e. pointwise.

3.4 The Cauchy transform in $L^2(\mathbb{R})$. The holomorphic Hardy space $\mathcal{H}^2(\Pi)$

As mentioned above, the map $f \mapsto P[f]$ is a one-to-one map between $L^2(\mathbb{R})$ and the space $h^2(\Pi)$ of harmonic functions $u(x, t)$ in the upper half space Π such that

$$\sup_t \int_{\mathbb{R}} |u(x, t)|^2 dx < +\infty.$$

The closed subspace of $h^2(\Pi)$ consisting of holomorphic functions is denoted by $\mathcal{H}^2(\Pi)$. From the decomposition

$$P[f](z) = \int_0^{+\infty} \hat{f}(\xi) e^{2\pi i z \xi} d\xi + \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z} \xi} d\xi$$

it follows that $P[f]$ is holomorphic if and only if \hat{f} is supported in $(0, +\infty)$ in which case

$$P[f](z) = \int_0^{+\infty} \hat{f}(\xi) e^{2\pi i z \xi} d\xi = Cf(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(y)}{y - z}$$

and is antiholomorphic (conjugate of an holomorphic function) if and only if $g = \hat{f}$ is supported in $(-\infty, 0)$.

From the relation $Cf = \frac{1}{2}(P[f] + iQ[f])$ we see that

$$\lim_{t \rightarrow 0} Cf(x + it) = \frac{1}{2}(f(x) + iHf(x)),$$

both in the L^2 - sense and pointwise a.e.

We may think then that on the Fourier transform side $L^2(0, +\infty)$ is the *holomorphic* closed subspace of $L^2(\mathbb{R})$, its orthogonal complement being of course $L^2(-\infty, 0)$. The orthogonal projection A onto this subspace, that in the Fourier transform side is simply defined $\widehat{Af}(\xi) = \hat{f}(\xi)1_{(0, +\infty)}(\xi)$ is done on the other side by A equal to the boundary value of Cf , that is, $A = \frac{1}{2}(I + iH)$. Indeed the multiplier of A is $\frac{1}{2}(1 + \text{sign}(\xi)) = 1_{(0, \infty)}$. The properties of H translate precisely in A being a projector, for instance

$$A^2 = \frac{1}{4}(I + iH)^2 = \frac{1}{4}(I - H^2 + 2iH) = \frac{1}{2}(I + iH) = A,$$

and similarly the others.

It follows from all this that the operators H, A and C are equivalent, and their properties the same.

Note finally that from the computations in section 3.2 it follows that the jump of the Cauchy transform

$$\lim_{t \rightarrow 0} Cf(x + it) - Cf(x - it) = f(x), a.e.x.$$

This is a general fact for Cauchy transforms. If Γ is an arbitrary rectifiable curve and

$$C_\Gamma f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z} dw, z \notin \Gamma,$$

then the boundary value at one point $w_0 \in \Gamma$ is

$$\frac{1}{2}(f(w_0) \pm i \frac{1}{\pi} p.v. \int_\Gamma \frac{f(w)}{w-w_0} dw),$$

and so the jump is $f(w_0)$. This result is known as Plemelj formula.

3.5 The Calderon-Zygmund decomposition

In chapter 2, in the setting of interpolation theory, and in a general measure space (X, μ) , we have used the following decomposition of a function $f \in L^1(X)$, given $\lambda > 0$,

$$f = g + b, g = f1_{|f| < \lambda}, b = f1_{|f| \geq \lambda}.$$

The function g (standing for *good*) inherits the property of f , meaning that $\|g\|_1 \leq \|f\|_1$, but has an additional good property, namely it is bounded by λ . The function b , which stands for *bad*, also is integrable with $\|b\|_1 \leq \|f\|_1$, and has an additional property, namely that its support has measure less than $\frac{\|f\|_1}{\lambda}$.

In measure spaces with more structure, as in the real line, it turns out that something more deep can be stated. This is the contents of a famous decomposition due to Calderon and Zygmund, that plays an important role in the theory of singular integrals. We need first another famous result, due to Riesz, named the Rising sun lemma, because its proof consists simply in thinking that the sun rises from the right of the graphic of g .

Lemma 1 (The rising sun lemma). Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with $g(-\infty) = +\infty, g(+\infty) = -\infty$. Let

$$E = \{x : \text{there exists } y > x \text{ such that } g(y) > g(x)\}.$$

Then E is open, and if not empty it is the union of intervals $I_k = (\alpha_k, \beta_k)$, with $g(\alpha_k) = g(\beta_k)$.

Theorem 4 (Calderon-Zygmund decomposition in \mathbb{R}). Let $f \in L^1(\mathbb{R}), f \geq 0, \lambda \geq 0$. Then, there is a sequence I_k of disjoint open intervals such that

$$\frac{1}{|I_k|} \int_{I_k} f = \lambda, f(x) \leq \lambda \text{ for a.e. } x \notin \cup_k I_k.$$

In particular,

$$\sum_k |I_k| = \frac{1}{\lambda} \int_{\cup I_k} f \leq \frac{1}{\lambda} \|f\|_1.$$

Proof. We apply the rising sun lemma to

$$g(x) = \int_{-\infty}^x f(t) dt - \lambda x.$$

Then $g(\alpha_k) = g(\beta_k)$ means exactly that the mean over I_k is λ . Next, by Lebesgue's differentiation theorem (chapter 1), we know that almost every point x has derivative $f(x) - \lambda$. Since at the points of E where $g'(x)$ exists it must be ≤ 0 , we have that $f(x) \leq \lambda$ a.e. outside E . \square

Then we define

$$g(x) = \begin{cases} f(x), & x \notin E \\ \lambda, & x \in E \end{cases} \quad b(x) = \begin{cases} 0, & x \notin E \\ f(x) - \lambda, & x \in E \end{cases}$$

Then $\|g\|_1 = \|f\|_1$, g is bounded by λ a.e. We have too that b is supported in a set $E = \cup_k I_k$ with measure

$$\sum |I_k| = \frac{1}{\lambda} \int_E f \leq \frac{1}{\lambda} \|f\|_1,$$

and also

$$\|b\|_1 = \sum_k \int_{I_k} |f(x) - \lambda| \leq 2 \sum_k \int_{I_k} f = 2\|f\|_1.$$

What's new with respect the previous "trivial" decomposition? The fact that we can write $b = \sum_k b_k$, with each b_k supported in I_k , with mean zero.

We refer to the decomposition $f = g + b$ as the Calderon-Zygmund decomposition at level λ .

3.6 Kolmogorov's and Riesz's theorem

The Kolmogorov's theorem asserts that the Hilbert transform satisfies a weak $(1, 1)$ estimate. He proved this using complex analysis. Here we shall give the real analysis proof based on the Calderon-Zygmund decomposition, since it goes over to dimension $d > 1$.

Theorem 5 (Kolmogorov's theorem). There exists a constant C such that for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$|\{x : |Hf(x)| > \lambda\}| \leq \frac{C}{\lambda} \|f\|_1.$$

Proof. We may assume that $f \geq 0$ by decomposing f in its real and imaginary part, $f = f_1 + if_2$, and then in positive and negative parts, $f_j =$

$f_j^+ - f_j^-$, $j = 1, 2$. We fix $\lambda > 0$, apply the Calderon-Zygmund decomposition at level λ and consider $f = g + b$. Then $g \in L^1 \cap L^2$ (because it is in L^1 and bounded), and so is b . We have $|Hf| \leq |Hg| + |Hb|$, whence

$$|\{|Hf| > \lambda\}| \leq |\{|Hg| > \frac{\lambda}{2}\}| + |\{|Hb| > \frac{\lambda}{2}\}|.$$

The first term is estimated using that H is bounded in L^2 : by Tchebychev's inequality, the first term is bounded by

$$\frac{4}{\lambda^2} \|Hg\|_2 = \frac{4}{\lambda^2} \|g\|_2.$$

But g being bounded by λ implies $\|g\|_2 \leq \lambda \|g\|_1 \leq \lambda \|f\|_1$, therefore this first term satisfies the required estimate. Let us now denote by I_k^* the interval with the same center c_k as I_k but twice its length, and set $E^* = \cup_k I_k^*$. Since

$$|E^*| \leq 2 \sum_k |I_k| \leq \frac{2}{\lambda} \|f\|_1,$$

we need just to estimate

$$|\{x \notin E^* : |Hb(x)| > \frac{\lambda}{2}\}|.$$

By Tchebychev's inequality once again, it is enough to prove that

$$\int_{\mathbb{R} \setminus E^*} |Hb(x)| dx \leq C \|f\|_1,$$

and for that, using $b = \sum b_k$, $Hb = \sum_k Hb_k$, it is in turn enough to prove that

$$\sum_k \int_{\mathbb{R} \setminus I_k^*} |Hb_k(x)| dx \leq C \|f\|_1.$$

With this purpose we will obtain a pointwise crucial estimate. For $x \notin I_k^*$,

$$Hb_k(x) = \frac{1}{\pi} \int_{I_k} \frac{b_k(y)}{x-y} dy.$$

But b_k has mean zero over I_k and therefore

$$Hb_k(x) = \frac{1}{\pi} \int_{I_k} b_k(y) \left(\frac{1}{x-y} - \frac{1}{x-c_k} \right) dy = \frac{1}{\pi} \int_{I_k} b_k(y) \left(\frac{y-c_k}{(x-y)(x-c_k)} \right) dy$$

where c_k denotes the center of I_k . Having exploited cancelation we now estimate the size of $|Hb_k(x)|$, $x \notin I_k^*$, using $|y-c_k| \leq |I_k|$, and that $\frac{1}{2}|x-c_k| \leq |x-y| \leq \frac{3}{2}|x-c_k|$ for $y \in I_k$, $x \notin I_k^*$ by

$$|Hb_k(x)| \leq C \frac{|I_k|}{|x-c_k|^2} \int_{I_k} |b_k(y)| dy, x \notin I_k^*.$$

Therefore,

$$\int_{\mathbb{R} \setminus I_k^*} |Hb_k(x)| dx \leq C|I_k| \left(\int_{I_k} |b_k(y)| dy \right) \int_{\mathbb{R} \setminus I_k^*} \frac{1}{|x - c_k|^2} dx.$$

But this last integral is a constant times $\frac{1}{|I_k|}$, so we see that

$$\int_{\mathbb{R} \setminus I_k^*} |Hb_k(x)| dx \leq \int_{I_k} |b_k(y)| dy,$$

so that summing in k we are done because $\|b\| \leq C\|f\|_1$. □

Theorem 6 (Riesz's theorem). The Hilbert transform satisfies a strong (p, p) -estimate for $1 < p < +\infty$:

$$\|Hf\|_p \leq C_p \|f\|_p, f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R}).$$

Proof. For $1 < p < 2$ this follows from Marcinkiewicz interpolation theorem. For $p > 2$ we argue by duality exploiting that H is essentially self-adjoint: for $f \in \mathcal{S}(\mathbb{R}^\Gamma)$,

$$\begin{aligned} \|Hf\|_p &= \sup_{\|g\|_{p'} \leq 1} \left| \int (Hf)g \right| = \\ &= \sup_{\|g\|_{p'} \leq 1} \left| \int f(Hg) \right| \leq \|f\|_p \|Hg\|_{p'} \leq \|f\|_p C_{p'} \|g\|_{p'} \leq C_{p'} \|f\|_p. \end{aligned}$$

□