Lecture notes, Singular Integrals

Joaquim Bruna, UAB

May 18, 2016

Chapter 3

The Hilbert transform

In this chapter we will study the Hilbert transform. This is a specially important operator for several reasons:

- Because of its relationship with summability for Fourier integrals in L^p-norms.
- Because it constitutes a link between real and complex analysis
- Because it is a *model case* for the general theory of singular integral operators.

A main keyword in the theory of singular integrals and in analysis in general is *cancellation*. We begin with some easy examples of what this means.

3.1 Some objects that exist due to cancellation

One main example of something that exists due to cancellation is the Fourier transform of functions in $L^2(\mathbb{R}^d)$. In fact the whole L^2-theory of the Fourier transform exists thanks to cancellation properties. Let us review for example the well-known Parseval's theorem, stating that for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ one has $\|\hat{f}\|_2 = \|f\|_2$, a result that allows to extend the definition of the Fourier transform to $L^2(\mathbb{R}^d)$. Formally,

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \, d\xi = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)\overline{f(y)}(\int_{\mathbb{R}^d} e^{2\pi i \xi \cdot (y-x)} d\xi) dx dy.$$

This being equal to $\int_{\mathbb{R}^d} |f(x)|^2 \, dx$ means formally that

$$\int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} \, d\xi = \delta_0(x).$$

Along the same lines, let us look at the Fourier inversion theorem, stating that whenever $f \in L^1(\mathbb{R}^d)$, $\hat{f} \in L^1(\mathbb{R}^d)$ one has

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi.$$

Again, the right hand side, by a formal use of Fubini's theorem becomes

$$\int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} f(y) e^{-2\pi i \xi \cdot y}\, dy) e^{2\pi i \xi \cdot x}\, d\xi = \int_{\mathbb{R}^d} f(y) (\int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot (y-x)}\, dy)\, d\xi.$$

If this is to be equal to $f(x)$ we arrive to the same formal conclusion, namely that superposition of all frequencies is zero outside zero. An intuitive way to understand this is by noting that $(d = 1)$

$$\int_R^R e^{2\pi i \xi x}\, d\xi = \frac{\sin(2\pi R x)}{\pi x},$$

is zero for x an integer multiple of $\frac{1}{R}$, so when $R \to +\infty$ the zeros become more and more dense.

The Fourier transform exists trivially if $f \in L^1(\mathbb{R}^d)$, but not so trivially if $f \in L^2(\mathbb{R}^d)$. It exists, in the L^2-sense

$$\hat{f}(\xi) = \lim_n \int_{\mathbb{R}} f_n(x) e^{-2\pi i x \xi}\, dx,$$

with f_n arbitrary in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that $f_n \to f$ in $L^2(\mathbb{R}^d)$. The most trivial choice is to take $f_R(x)$ to be equal to $f(x)$ if $|x| < R$ and zero elsewhere, so that

$$\hat{f}(\xi) = \lim_R \int_{-R}^R f(x) e^{-2\pi i x \xi}\, dx,$$

the limit being in the L^2-sense. It is a general and very deep result due to L. Carleson that this limits exists a.e.

To illustrate all this we will consider the Fourier transform of two particular functions, none of which in $L^1(\mathbb{R})$, and study the pointwise behaviour of the limit above.

For instance, $\int_{\mathbb{R}} \frac{\sin x}{x}\, dx$ is conditionally convergent, meaning that

$$\lim_{R \to \infty} \int_{-R}^R \frac{\sin x}{x}\, dx \tag{3.1}$$

exists while $\int_{\mathbb{R}} |\frac{\sin x}{x}|\, dx = +\infty$. The later can be proved by estimating from below the contribution to the integral at points $|x - (\frac{\pi}{2} + k\pi)| < \delta, k \in \mathbb{Z}$. Alternatively, noting that the Fourier transform of $1_{[-a,a]}$ is

$$\int_a^a e^{-2\pi x \xi}\, dx = \frac{\sin 2\pi a \xi}{\pi \xi},$$

we can argue that it cannot be integrable, because if it were, by Fourier inversion theorem, $1_{[-a,a]}$ would be continuous. We note too that by Parseval's theorem

$$\int_{\mathbb{R}} (\frac{\sin 2\pi a\xi}{\pi\xi})^2 d\xi = \int_{\mathbb{R}} 1^2_{[-a,a]} = 2a.$$

Regarding the value of the integral, we will see that the limit

$$\lim_{R\to+\infty} \int_{-R}^{R} \frac{\sin 2\pi a\xi}{\pi\xi} e^{2\pi i\xi x} d\xi = 1_{[-a,a]}(x), \tag{3.2}$$

that as we know exists in the L^2 sense, exists too pointwise a.e., in fact for $x \neq \pm a$. Indeed, by using residues we have that

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{-R}^{R} \frac{e^{i\alpha\xi}}{\xi} d\xi$$

equals $\frac{1}{2} Res(\frac{e^{i\alpha\xi}}{\xi}, 0) = \frac{1}{2}$ if $\alpha > 0$ and $-\frac{1}{2}$ if $\alpha < 0$, and this implies (3.2). For $x = 0$ we get that

$$\lim_{R\to+\infty} \int_{-R}^{R} \frac{\sin 2\pi a\xi}{\pi\xi} d\xi = 1,$$

from which it follows that the integral in (3.1) equals π.

Another example are the Fresnel integrals

$$\int_{\mathbb{R}} \sin x^2 dx, \int_{R} \cos x^2 dx.$$

By the change of variable $u = x^2$ one has

$$\int_0^R \sin x^2 dx = \frac{1}{2} \int_0^{R^2} \frac{\sin u}{\sqrt{u}} du,$$

so the integral is not absolutely convergent. To compute the integral we will see more generally that

$$\lim_{R\to+\infty} \int_{-R}^{R} e^{-\alpha\pi x^2} e^{-2\pi i x\xi} dx = 2 \lim_{R\to+\infty} \int_0^R e^{-\alpha\pi x^2} \cos 2\pi x\xi \, dx = \frac{1}{\sqrt{\alpha}} e^{-\frac{\pi}{\alpha}\xi^2}. \tag{3.3}$$

for $\operatorname{Re}\alpha \geq 0$ and all $\xi \in \mathbb{C}$. For $\operatorname{Re}\alpha > 0$, the above is the Fourier transform of $f_\alpha(x) = e^{-\alpha\pi x^2}$, a function in the Schwarz class. Since it is holomorphic in α and for α, ξ real equals $\frac{1}{\sqrt{\alpha}} e^{-\frac{\pi}{\alpha}\xi^2}$, the same holds true by analytic continuation first in α and then in ξ, that is

$$F(\alpha, \xi) = \int_{\mathbb{R}} e^{-\alpha\pi x^2} e^{-2\pi i x\xi} dx = \frac{1}{\sqrt{\alpha}} e^{-\frac{\pi}{\alpha}\xi^2}, \operatorname{Re}\alpha > 0, \xi \in \mathbb{C}.$$

By making $\operatorname{Re}\alpha \to 0$ we can already conclude that the Fourier transform, in the sense of distributions, of $e^{-\alpha\pi x^2}$ is $\frac{1}{\sqrt{\alpha}}e^{-\frac{\pi}{\alpha}\xi^2}$ also if $\operatorname{Re}\alpha = 0$. However, this does not imply (3.3). To compute the limit in (3.3) when $\operatorname{Re}\alpha = 0$ we will exploit the analyticity in x as follows. We consider the contour in the z-plane given by the segment $[0, R]$, the arc $t \mapsto Re^{i\pi t}, 0 \le t \le \frac{1}{4}$, and the segment from $Re^{i\frac{\pi}{4}}$ to 0, and apply the Cauchy theorem to the entire function $e^{-\alpha\pi z^2}\cos(2\pi z\xi)$. The contribution of the arc is bounded by aRe^{-bR^2+cR} with some positive constants a, b, c depending on ξ, and so has limit zero when $R \to \infty$. Hence, parametrizing with $z = t\frac{1+i}{\sqrt{2}}$, we get

$$\lim_{R\to+\infty}\int_0^R e^{-\alpha\pi x^2}\cos(2\pi x\xi)\,dx = \frac{1+i}{\sqrt{2}}\lim_{R\to\infty}\int_0^R e^{-\alpha\pi i t^2}\cos(2\pi\frac{1+i}{\sqrt{2}}t\xi)dt.$$

If $\operatorname{Re} i\alpha > 0$, that is, $\operatorname{Im}\alpha < 0$ this shows that the limits exist and equals

$$\frac{1+i}{\sqrt{2}}F(i\alpha, \frac{1+i}{\sqrt{2}}\xi) = \frac{1}{\sqrt{\alpha}}e^{-\frac{\pi}{\alpha}\xi^2},$$

as claimed. For $\xi = 0$ and $\alpha = -\frac{i}{\pi}$ we get the value $\frac{\sqrt{\pi}}{2\sqrt{2}}$ for the Fresnel integrals.

3.2 The conjugate Poisson kernel and the Cauchy transform

One method to study the summability of the Fourier integral is by the means method. The Abel method consists in using

$$\int_{\mathbb{R}^d}\hat{f}(\xi)e^{-2\pi t|\xi|}e^{2\pi i x\cdot\xi}\,d\xi.$$

The behaviour as $t \to 0$ is easily understood. Indeed, by Fubini's theorem the above equals

$$\int_{\mathbb{R}^d} f(y)(\int_{\mathbb{R}^d} e^{-2\pi t|\xi|}e^{2\pi i\xi(x-y)}\,d\xi)dy = (f * P_t)(x),$$

with

$$P_t(x) = \int_{\mathbb{R}^d} e^{-2\pi t|\xi|}e^{2\pi i x\cdot\xi}\,d\xi.$$

Let us compute explicitly P_t in $d = 1$:

$$P_t(x) = \int_0^{+\infty} e^{2\pi\xi(ix-t)}\,d\xi + \int_{-\infty}^0 e^{2\pi\xi(ix+t)}\,d\xi = A + B,$$

with

$$A = \frac{1}{2\pi(t-ix)}, B = \overline{A} = \frac{1}{2\pi(t+ix)}.$$

This gives

$$P_t(x) = \frac{1}{\pi} \frac{t}{x^2 + t^2}.$$

Note that $P_t(x) = \frac{1}{t} P(x/t)$, with $P = P_1$, and that $P_1, e^{-2\pi|\xi|}$ are a Fourier pair, P is positive with integral one. In dimension $d > 1$ the computation is harder, the result being $P_t(x) = t^{-d} P(x/t)$, with

$$P(x) = c_d \frac{1}{(|x|^2 + 1)^{\frac{d+1}{2}}}, c_d = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}},$$

It is convenient to look as $(f * P_t)(x)$ as a function of (x,t) in the upper half space $t > 0$. Noticing that P is in all L^p-spaces we define

Definition 1. For $f \in L^p(\mathbb{R}^d), 1 \leq p \leq +\infty$, we define the *Poisson transform* $u = P[f]$ as the function defined in the upper half-space $t > 0$ by

$$P[f](x,t) = (f * P_t)(x) = \frac{1}{\pi} \int_{\mathbb{R}^d} f(y) \frac{t}{(x-y)^2 + t^2} \, dy = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{-2\pi t|\xi|} e^{2\pi i x\cdot\xi} \, d\xi$$

Since we will need it again in a more general form we state again the main properties of convolutions with functions of the form $h_t(x) = t^{-d} h(x/t)$, with $h \in \mathrm{L}^1(\mathbb{R}^d)$.

Proposition 1. Let $h \in L^1(\mathbb{R}^d), m = \int h$, and set $h_t(x) = t^{-d} h(x/t)$.. For $f \in L^p(\mathbb{R}^d), 1 \leq p \leq +\infty$, consider $f * h_t$. Then

- $\|f * h_t\|_p \leq \|f\|_p \|h\|_1$, and $f * h_t \to mf$ as $t \to 0$ in $L^p(\mathbb{R}^d), 1 \leq p < +\infty$.
- If f is bounded and continuous at x_0 then $f * h_t(x_0) \to mf(x_0)$.
- If f is bounded and uniformly continuous then $f * h_t \to mf$ uniformly.
- If the *least radially decreasing majorant* of h,

  $$\Psi(x) = \sup_{|y| \geq |x|} |h(x)|,$$

  is in $L^1(\mathbb{R}^d)$, then $f * h_t(x) \to mf(x)$ a.e.

Proof. The continuous Minkowski inequality implies the first assertion. We write

$$f*h_t(x) = \int_{\mathbb{R}^d} f(x-y) h_t(y) dy, f*h_t(x) - mf(x) = \int_{\mathbb{R}^d} (f(x-y) - f(x)) h_t(y) dy,$$

so that

$$|f * h_t(x) - mf(x)| \leq \int_{\mathbb{R}^d} |f(x - ty) - f(x)| |h(y)| dy,$$

and by continuous Minkowski inequality again

$$\|f * h_t - mf\|_p \leq \int_{\mathbb{R}^d} \|\tau_{ty} f - f\|_p |h(y)| dy.$$

The dominated convergence theorem implies then all assertions but the last. We saw in the first chapter that

$$\sup_t |f * h_t(x)| \leq \|\Psi\|_1 Mf(x),$$

so that the maximal function on the left satisfies a weak $(1, 1)$ estimate, hence it suffices to check the last assertion for f continuous with compact support, and this is the second statement. □

Theorem 1. Let $f \in L^p(\mathbb{R}^d), 1 \leq p \leq +\infty$ and let $u = P[f]$ be its Poisson transform. Then

- The function u is harmonic in the upper half-space $t > 0$.
- The function u satisfies

$$\sup_t \left(\int_{\mathbb{R}^d} |u(x,t)|^p \, dt \right)^{\frac{1}{p}} \leq \|f\|_p, 1 \leq p < \infty, |u(x,t)| \leq \|f\|_\infty, p = +\infty. \tag{3.4}$$

  and $P_t f \to f$ in $L^p, 1 \leq p < \infty$.

- The maximal function

$$P^* f(x) = \sup_t |P_t f(x)$$

  satisfies $P^* f \leq CMf(x)$, where Mf denotes the Hardy-Littlewood maximal function of f. The same holds for the non-tangential maximal function

$$P^*_{nt}(x) = \sup_{(y,t), |y-x|<ct} |P_t f(y)|.$$

- $\lim_{t \to 0} u(x,t) = f(x)$ a.e. In fact the non-tangential limit

$$\lim_{(y,t) \to x, |y-x| \leq ct} u(y,t)$$

  is a.e. equal to $f(x)$.

The fact that u is harmonic is checked on the Fourier transform side definition. The other properties are restatements of proposition 1 in the particular case of the Poisson kernel. The last statement is checked first for f continuous with compact support and then it follows from the fact that the non-tangential maximal function is also bounded by Mf for general f.

The space of harmonic functions in the half-space satisfying (3.4) is denoted by $h^p(\mathbb{R}^{d+1}_+)$. It is possible, for $1 < p \leq +\infty$ and using the Helly selection theorem (that is Banach-Alaouglu theorem for L^p- spaces) to prove that it is exactly the space of harmonic functions that are Poisson transforms of functions in $L^p(\mathbb{R}^d)$. For $p = 1$ it is the space of Poisson transforms $P[\mu](x,t) = (\mu * P_t)(x)$ of finite complex Borel measures μ in $\mathbb{R}^d$. If $u \in h^1$, and moreover, $\lim_t u(\cdot, t) = f$ exists in L^1, then $u = P[f]$.

We now consider only $d = 1$. We know that an harmonic function u in a simply connected domain has up to constants a unique harmonic conjugate v, that is, such that $u + iv$ is holomorphic. Let us compute explicitly the harmonic conjugate of $u = P[f]$. For this purpose we rewrite, in terms of $z = x + it$,

$$u(z) = P[f](z) = \int_0^{+\infty} \hat{f}(\xi) e^{2\pi i z \xi} \, d\xi + \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z} \xi} \, d\xi$$

which exhibits $P[f]$ as the sum of an holomorphic function and a antiholomorphic function.

We define $v(z)$ by setting

$$iv(z) = \int_0^{+\infty} \hat{f}(\xi) e^{2\pi i z \xi} \, d\xi - \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z} \xi} \, d\xi,$$

so that v is also harmonic and

$$(u + iv)(z) = 2 \int_0^{+\infty} \hat{f}(\xi) e^{2\pi i z \xi} \, d\xi,$$

is indeed holomorphic.

This is the expression of v in the Fourier transform side. On the other side we can use the same computations done above. Indeed, to define iv we have separated the contribution of positive and negative ξ and changed the sign of the later. This corresponds to considering, in the above notations, $A - B = A - \overline{A}$ instead of $A + B$. Since

$$A - \overline{A} = \frac{1}{2\pi} \left(\frac{1}{t - ix} - \frac{1}{t + ix} \right) = \frac{i}{\pi} \frac{x}{x^2 + t^2},$$

this means that $v(x,t) = (f * Q_t)(x)$, where

$$Q_t(x) = \frac{1}{\pi} \frac{x}{x^2 + t^2}, \quad \widehat{Q_t}(\xi) = -i \, \mathrm{sign}(\xi) \, e^{-2\pi t |\xi|}.$$

Note that $Q_t(x) = \frac{1}{t} Q(x/t)$, with $Q(x) = \frac{1}{\pi} \frac{x}{x^2+1}$ and that

$$P_t(x) + iQ_t(x) = \frac{1}{\pi} \frac{t + ix}{x^2 + t^2} = \frac{i}{\pi z}.$$

The family Q_t is called the *conjugate Poisson kernel.* Notice that $Q \notin L^1(\mathbb{R})$ but $Q \in \mathrm{L}^q(\mathbb{R})$ for $1 < q \leq +\infty$.

Definition 2. If $f \in L^p(\mathbb{R}), 1 \le p < +\infty$ we define the *conjugate Poisson transform* $Q[f]$ of f as the harmonic function in the upper half-plane defined by

$$Q[f](z) = (f * Q_t)(x) = \frac{1}{\pi}\int_{\mathbb{R}} f(x-y)\frac{y}{y^2+t^2}\,dy, z = x+it, t>0,$$

and the *Cauchy transform* by

$$Cf(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(y)}{y-z}\,dy, z \notin \mathbb{R}.$$

The computations above show that

$$Cf(z) = \frac{1}{2}(P[f] + iQ[f])(z) = \int_0^{+\infty} \hat{f}(\xi)e^{2\pi i z\xi}\,d\xi, t = \operatorname{Im} z > 0.$$

They also show that

$$(P[f] - iQ[f])(z) = 2\int_{-\infty}^0 \hat{f}(\xi)e^{2\pi i\bar{z}\xi}\,d\xi = -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(y)}{y-\overline{z}}\,dy,$$

or

$$Cf(z) = \frac{1}{2}(-P[f](\overline{z}) + iQ[f](\overline{z})), \operatorname{Im} z < 0.$$

3.3 The Hilbert transform

We want to study now the behaviour of $f * Q_t$ as $t \to 0$. Formally we have

$$\lim_{t\to 0} Q_t(x) = \frac{1}{\pi}\frac{1}{x}, \lim_{t\to 0}\widehat{Q_t}(\xi) = -i\,\mathrm{sign}(\xi).$$

Convolution with $\frac{1}{x}$ does not mean sense since it is not integrable and neither is a distribution. Now, recall that thanks to Parseval's theorem, any bounded measurable function m in $\mathbb{R}$ defines a translation invariant operator T in $L^2(\mathbb{R})$ by setting $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$, and then $\|T\| = \|m\|_\infty$. In our case, then, it makes sense

Definition 3. The Hilbert transform H is the translation-invariant operator defined in $L^2(\mathbb{R})$ by

$$\widehat{Hf}(\xi) = -i\mathrm{sign}(\xi)\hat{f}(\xi),$$

which amounts to

$$Hf(x) = -i\lim_{R\to+\infty}\int_R^R sign(\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi, \text{ in } L^2,$$

or, equivalently,

$$Hf = \lim_{t\to 0} f * Q_t, \text{ in } L^2(\mathbb{R}).$$

Theorem 2. The Hilbert transform H satisfies:

- It is an isometry of $L^2(\mathbb{R})$ with $H^* = H^{-1} = -H$, hence
$$\int Hfg = -\int fHg, f, g \in L^2(\mathbb{R}).$$

- With the notation $\rho_\lambda f(x) = f(\lambda x), \lambda > 0$ for positive dilations and $\tilde{f}(x) = f(-x)$ for the reflection operator, H commutes with the $\rho_\lambda, H\rho_\lambda = \rho_\lambda H$, and anticommutes with the reflection operator, $H(\tilde{f})(x) = -\widetilde{Hf}(x) = -Hf(-x)$.

- If T is a bounded operator in $L^2(\mathbb{R})$ that commutes with translations, with positive dilations, and anticommutes with the reflection operator, then $T = cH$ for some constant c.

Proof. Let us call $m(\xi) = -i\,\mathrm{sign}(\xi)$ for the multiplier of H. The first statement then corresponds to $|m| = 1, \overline{m} = m^{-1} = -m$. Now, one has $\widehat{\rho_\lambda f}(\xi) = \hat{f}(\xi/\lambda)$, and $\hat{\tilde{f}}(\xi) = \hat{f}(-\xi) = \tilde{\hat{f}}(\xi)$; therefore, a a translation invariant operator with multiplier m commutes with positive dilations iff m is a constant on $\mathbb{R}^+$, and it anticommutes with the reflection iff m is odd. Thus, H has these properties and any other operator with the same properties is a scalar multiple of H. □

Notice that the same argument shows that the multiples of the identity operator are the only ones that commute with translations, with positive dilations and with the reflection.

Thus we have a clear picture of what is $\lim_t Q_t$ on the Fourier transform side and now we would like to understand it as a convolution operator. Formally,
$$\lim_t Q_t(x) = \frac{1}{\pi}\frac{1}{x},$$
which is not a locally integrable function, neither it is a distribution, it does not make sense something like
$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{y-x}\,dy.$$

At this point we recall the definition of the so called its *principal value*, namely,
$$\lim_{t\to 0}\frac{1}{\pi}\int_{|y-x|>t}\frac{f(y)}{x-y}\,dy = \frac{1}{\pi}\int_{|y|>t}\frac{f(x-y)}{y}\,dy.$$

We will set $\Phi_t(x) = \frac{1}{x}1_{|x|>t}$, and
$$H_t f(x) = \frac{1}{\pi}(f * \Phi_t)(x) = \frac{1}{\pi}\int_{|y|>t}\frac{f(x-y)}{y}\,dy.$$

The functions Φ_t, that also obey $\Phi_t(x) = \frac{1}{t}\Phi(x/t)$, with $\Phi = \Phi_1$, have limit in the distribution sense, that is, the limit

$$\lim_{t\to 0}\langle \Phi_t, \varphi\rangle = \lim_{t\to 0}\int_{|y|>t} \frac{\varphi(y)}{y}\, dy,$$

exists for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$. This is because exploiting that the kernel is odd the limit equals

$$\int_0^{+\infty} \frac{\varphi(y) - \varphi(-y)}{y}\, dy = \int_{|y|<1} \frac{\varphi(y) - \varphi(0)}{y}\, dy + \int_{|y|>1} \frac{\varphi(y)}{y}\, dy.$$

The limit of Φ_t in the distribution sense is called $p.v\frac{1}{x}$. Thus,

$$\lim_{t\to 0} H_t\varphi(x) = \frac{1}{\pi}\varphi*(p.v.\frac{1}{x}) = \lim_{t\to 0}\frac{1}{\pi}\int_{|y|>t} \frac{\varphi(x-y)}{y}\, dy = \int_0^{\infty} \frac{\varphi(x-y) - \varphi(x+y)}{y}\, dy,$$

exists for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and all x. Now we will see that in fact it exists for general $\varphi \in L^2$, and equals $H(\varphi)$.

Theorem 3. For $f \in L^2(\mathbb{R})$, the Hilbert transform, that has been defined as

$$Hf = \lim_{t\to 0} f * Q_t, \text{ in } L^2(\mathbb{R}),$$

or equivalently, by $\widehat{Hf}(\xi) = -i\, sign(\xi)\, \hat{f}(\xi)$, is also given by

$$Hf = \frac{1}{\pi}(f * p.v.\frac{1}{x}) = \lim_{t\to 0} H_t f = \lim_{t\to 0}\frac{1}{\pi} f * \Phi_t,$$

that is

$$Hf(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(x-y)}{y}\, dy,$$

where this limit is in L^2-norm as well. Moreover, for both expressions

$$\lim_{t\to 0} f * Q_t(x) = \lim_{t\to 0} H_t f(x) = Hf(x) \ a.e.x.$$

Proof. First note that for $f \in L^2(\mathbb{R})$ we have that $Q_t f = P_t Hf$ because both have the same Fourier transform $-isign(\xi)e^{-2\pi t|\xi|}\hat{f}(\xi)$. Since P_t is an approximation of the identity with a radial function, it follows that $Q_t f(x) = P_t(Hf)(x)$ has limit $Hf(x)$ a.e. Next we look at the difference $h_t(x) = Q_t(x) - \frac{1}{\pi}\Phi_t(x)$, that also obeys the rule $h_t(x) = \frac{1}{t}h(x/t)$,, with

$$h(x) = \frac{1}{\pi}\begin{cases} \frac{x}{1+x^2}, |x| < 1 \\ \frac{x}{1+x^2} - \frac{1}{x} = -\frac{1}{x(1+x^2)}, |x| > 1 \end{cases}.$$

Since h is odd, it has integral $m = \int h = 0$; moreover, its radial majorant Ψ is explicitly, up to the factor $\frac{1}{\pi}$, equal to $\frac{1}{|x|(1+|x|^2)}$ if $|x| > 1$ and so it is integrable. Applying the proposition 1, we see that $\lim_{t\to 0} f * h_t = 0$, for $f \in L^p(\mathbb{R}), 1 \leq p < +\infty$, both in L^p- norm and pointwise a.e. Since for $f \in L^2(\mathbb{R})$, the limit of $f * Q_t$ equals Hf both in L^2-norm and pointwise a.e., the same happens with $H_t f$. □

That completes the definition of H in $L^2(\mathbb{R})$, from three points of view: through a multiplier $-i\, sign(\xi)$ of $L^2(\mathbb{R})$, through a complex analysis point of view (limit of the conjugate harmonic function $Q[f]$), and from the real analysis point of view (as a principal value integral), the limits existing both in the L^2-sense and a.e. pointwise.

3.4 The Cauchy transform in $L^2(\mathbb{R})$. The holomorphic Hardy space $\mathcal{H}^2(\Pi)$

As mentioned above, the map $f \mapsto P[f]$ is a one-to-one map between $L^2(\mathbb{R})$ and the space $h^2(\Pi)$ of harmonic functions $u(x,t)$ in the upper half space Π such that

$$\sup_t \int_{\mathbb{R}} |u(x,t)|^2\, dx < +\infty.$$

The closed subspace of $h^2(\Pi)$ consisting of holomorphic functions is denoted by $\mathcal{H}^2(\Pi)$. From the decomposition

$$P[f](z) = \int_0^{+\infty} \hat{f}(\xi) e^{2\pi i z\xi}\, d\xi + \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z}\xi}\, d\xi$$

it follows that $P[f]$ is holomorphic if and only if $\hat{f}$ is supported in $(0, +\infty)$ in which case

$$P[f](z) = \int_0^{+\infty} \hat{f}(\xi) e^{2\pi i z\xi}\, d\xi = Cf(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(y)}{y - z}$$

and is antiholomorphic (conjugate of an holomorphic function) if and only if $g = \hat{f}$ is supported in $(-\infty, 0)$.

From the relation $Cf = \frac{1}{2}(P[f] + iQ[f])$ we see that

$$\lim_{t\to 0} Cf(x+it) = \frac{1}{2}(f(x) + iHf(x)),$$

both in the L^2- sense and pointwise a.e.

We may think then that on the Fourier transform side $L^2(0, +\infty)$ is the *holomorphic* closed subspace of $L^2(\mathbb{R})$, its orthogonal complement being of course $L^2(-\infty, 0)$. The orhogonal projection A onto this subspace, that in the Fourier transform side is simply defined $\widehat{Af}(\xi) = \hat{f}(\xi) 1_{0,+\infty}(\xi)$ is done on the other side by A equal to the boundary value of Cf, that is, $A = \frac{1}{2}(I + iH)$. Indeed the multiplier of A is $\frac{1}{2}(1 + sign(\xi)) = 1_{(0,\infty)}$. The properties of H translate precisely in A being a projector, for instance

$$A^2 = \frac{1}{4}(I + iH)^2 = \frac{1}{4}(I - H^2 + 2iH) = \frac{1}{2}(I + iH) = A,$$

and similarly the others.

It follows from all this that the operators H, A and C are equivalent, and their properties the same.

Note finally that from the computations in section 3.2 it follows that the jump of the Cauchy transform

$$\lim_{t\to 0} Cf(x + it) - Cf(x - it) = f(x), a.e.x.$$

This is a general fact for Cauchy transforms. If Γ is an arbitrary rectifiable curve and

$$C_\Gamma f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w - z}\, dw, z \notin \Gamma,$$

then the boundary value at one point $w_0 \in \Gamma$ is

$$\frac{1}{2}(f(w_0) \pm i\frac{1}{\pi}\, p.v. \int_\Gamma \frac{f(w)}{w - w_0}\, dw),$$

and so the jump is $f(w_0)$. This result is known as Plemej formula.

3.5 The Calderon-Zygmund decomposition

In chapter 2, in the setting of interpolation theory, and in a general measure space (X, μ), we have used the following decomposition of a function $f \in L^1(X)$, given $\lambda > 0$,

$$f = g + b, g = f1_{|f|<\lambda}, b = f1_{|f|\geq\lambda}.$$

The function g (standing for g) inherits the property of f, meaning that $\|g\|_1 \leq \|f\|_1$, but has an additional good property, namely it is bounded by λ. The function b, which stands for bad, also is integrable with $\|b\|_1 \leq \|f\|_1$, and has an additional property, namely that its support has measure less than $\frac{\|f\|_1}{\lambda}$.

In measure spaces with more structure, as in the real line, it turns out that something more deep can be stated. This is the contents of a famous decomposition due to Calderon and Zygmund, that plays an important role in the theory of singular integrals. We need first another famous result, due to Riesz, named the Rising sun lemma, because its proof consists simply in thinking that the sun rises from the right of the graphic of g.

Lemma 1 (The rising sun lemma)**.** Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function with $g(-\infty) = +\infty, g(+\infty) = -\infty$. Let

$$E = \{x : \text{there exists}\, y > x\, \text{such that}\, g(y) > g(x)\}.$$

Then E is open, and if not empty it is the union of intervals $I_k = (\alpha_k, \beta_k)$, with $g(\alpha_k) = g(\beta_k)$.

Theorem 4 (Calderon-Zygmund decomposition in $\mathbb{R}$)**.** Let $f \in L^1(\mathbb{R}), f \geq 0, \lambda \geq 0$. Then, there is a sequence I_k of disjoint open intervals such that

$$\frac{1}{|I_k|}\int_{I_k} f = \lambda,\ f(x) \leq \lambda\, \text{for a.e. }\ x \notin \cup_k I_k.$$

In particular,

$$\sum_k |I_k| = \frac{1}{\lambda}\int_{\cup I_k} f \leq \frac{1}{\lambda}\|f\|_1.$$

Proof. We apply the rising sun lemma to

$$g(x) = \int_{-\infty}^{x} f(t)\, dt - \lambda x.$$

Then $g(\alpha_k) = g(\beta_k)$ means exactly that the mean over I_k is λ. Next, by Lebesgue's differentiation theorem (chapter 1), we know that al almost every point g has derivative $f(x) - \lambda$. Since at the points of E where $g'(x)$ exists it must be ≤ 0, we have that $f(x) \leq \lambda$ a.e. outside E. □

Then we define

$$g(x) = \begin{cases} f(x), x \notin E \\ \lambda, x \in E \end{cases} \qquad b(x) = \begin{cases} 0, x \notin E \\ f(x) - \lambda, x \in E \end{cases}$$

Then $\|g\|_1 = \|f\|_1$, g is bounded by λ a.e. We have too that b is supported in a set $E = \cup_k I_k$ with measure

$$\sum |I_k| = \frac{1}{\lambda}\int_E f \leq \frac{1}{\lambda}\|f\|_1,$$

and also

$$\|b\|_1 = \sum_k \int_{I_k} |f(x) - \lambda| \leq 2\sum_k \int_{I_k} f = 2\|f\|_1.$$

What's new with respect the previous "trivial" decomposition? The fact that we can write $b = \sum_k b_k$, with each b_k supported in I_k, with mean zero.

We refer to the decomposition $f = g + b$ as the Calderon-Zygmund decomposition at level λ.

3.6 Kolmogorov's and Riesz's theorem

The Kolmogorov's theorem asserts that the Hilbert transform satisfies a weak $(1,1)$ estimate. He proved this using complex analysis. Here we shall give the real analysis proof based on the Calderon-Zygmund decomposition, since it goes over to dimension $d > 1$.

Theorem 5 (Kolmogorov's theorem)**.** There exists a constant C such that for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$|\{x : |Hf(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1.$$

Proof. We may assume that $f \geq 0$ by decomposing f in its real and imaginary part, $f = f_1 + if_2$, and then in positive and negative parts, $f_j = f_j^+ - f_j^-, j = 1, 2$. We fix $\lambda > 0$, apply the Calderon-Zygmund decomposition at level λ and consider $f = g + b$. Then $g \in L^1 \cap L^2$ (because it is in L^1 and bounded), and so is b. We have $|Hf| \leq |Hg| + |Hb|$, whence

$$|\{|Hf| > \lambda\}| \leq |\{|Hg| > \frac{\lambda}{2}\}| + |\{|Hb| > \frac{\lambda}{2}\}|.$$

The first term is estimated using that H is bounded in L^2: by Tchebychev's inequality, the first term is bounded by

$$\frac{4}{\lambda^2}\|Hg\|_2 = \frac{4}{\lambda^2}\|g\|_2.$$

But g being bounded by λ implies $\|g\|_2 \leq \lambda\|g\|_1 \leq \lambda\|f\|_1$, therefore this first term satisfies the required estimate. Let us now denote by I_k^* the interval with the same center c_k as I_k but twice its length, and set $E^* = \cup_k I_k^*$. Since

$$|E^*| \leq 2\sum_k |I_k| \leq \frac{2}{\lambda}\|f\|_1,$$

we need just to estimate

$$|\{x \notin E^* : |Hb(x)| > \frac{\lambda}{2}\}|.$$

By Tchebychev's inequality once again, it is enough to prove that

$$\int_{\mathbb{R}\setminus E^*} |Hb(x)|dx \leq C\|f\|_1,$$

and for that, using $b = \sum b_k, Hb = \sum_k Hb_k$, it is in turn enough to prove that

$$\sum_k \int_{\mathbb{R}\setminus I_k^*} |Hb_k(x)|\, dx \leq C\|f\|_1.$$

With this purpose we will obtain a pointwise crucial estimate. For $x \notin I_k^*$,

$$Hb_k(x) = \frac{1}{\pi} \int_{I_k} \frac{b_k(y)}{x-y}\, dy.$$

But b_k has mean zero over I_k and therefore

$$Hb_k(x) = \frac{1}{\pi} \int_{I_k} b_k(y) \left(\frac{1}{x-y} - \frac{1}{x-c_k} \right) dy = \frac{1}{\pi} \int_{I_k} b_k(y) \left(\frac{y-c_k}{(x-y)(x-c_k)} \right) dy$$

where c_k denotes the center of I_k. Having exploited cancelation we now estimate the size of $|Hb_k(x)|, x \notin I_k^*$, using $|y-c_k| \leq |I_k|$, and that $\frac{1}{2}|x-c_k| \leq |x-y| \leq \frac{3}{2}|x-c_k|$ for $y \in I_k, x \notin I_k^*$ by

$$|Hb_k(x)| \leq C \frac{|I_k|}{|x-c_k|^2} \int_{I_k} |b_k(y)|\, dy, x \notin I_k^*.$$

Therefore,

$$\int_{\mathbb{R}\setminus I_k^*} |Hb_k(x)|\, dx \leq C|I_k|(\int_{I_k} |b_k(y)|\, dy) \int_{\mathbb{R}\setminus I_k^*} \frac{1}{|x-c_k|^2}\, dx.$$

But this last integral is a constant times $\frac{1}{|I_k|}$, so we see that

$$\int_{\mathbb{R}\setminus I_k^*} |Hb_k(x)|\, dx \leq \int_{I_k} |b_k(y)|\, dy,$$

so that summing in k we are done because $\|b\| \leq C\|f\|_1$. □

Theorem 6 (Riesz's theorem)**.** The Hilbert transform satisfies a strong (p,p)-estimate for $1 < p < +\infty$:

$$\|Hf\|_p \leq C_p \|f\|_p, f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R}).$$

Proof. For $1 < p < 2$ this follows from Marcinkiewicz interpolation theorem. For $p > 2$ we argue by duality exploiting that H is essentially self-adjoint: for $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\|Hf\|_p = \sup_{\|g\|_{p'} \leq 1} \left| \int (Hf) g \right| =$$

$$= \sup_{\|g\|_{p'} \leq 1} \left| \int f(Hg) \right| \leq \|f\|_p \|Hg\|_{p'} \leq \|f\| C_{p'} \|g\|_{p'} \leq C_{p'} \|f\|_p.$$

□

3.7 The Hilbert transform in $L^p(\mathbb{R}), 1 \leq p < \infty$ and in $\mathcal{S}(\mathbb{R}^d)$

By the usual extension method based in density arguments, we may now extend the definition of H to $L^p(\mathbb{R}), 1 < p < +\infty$: if say $\varphi_n \in \mathcal{S}(\mathbb{R}^d)$ approach $f \in L^p(\mathbb{R})$ in the L^p-norm, we define $Hf = \lim_n H\varphi_n$. A slightly different consideration occurs if $f \in L^1(\mathbb{R})$. We may take again $\varphi_n \in \mathcal{S}(\mathbb{R}^d)$ approaching f in $L^1(\mathbb{R})$; then the weak $(1,1)$ estimate says that $H\varphi_n$ is a Cauchy sequence in measure, that is, for each $\varepsilon > 0$,

$$\lim_{n,m\to+\infty} |\{|H\varphi_n - H\varphi_m| > \varepsilon\}| = 0,$$

and hence Hf can be a.e. be defined as the limit in measure of the $H\varphi_n$, that is,

$$\lim_{n,\to+\infty} |\{|H\varphi_n - Hf| > \varepsilon\}| = 0.$$

The function Hf thus defined is in weak $L^1 = L^{1,\infty}(\mathbb{R})$, that is

$$|\{|Hf| > \lambda\}| \leq \frac{C}{\lambda}.$$

Alternatively, to define H in $L^p(\mathbb{R}), 1 \leq p < +\infty$, we may argue that H_t can be treated exactly as H so that it satisfies a strong (p,p) estimate, $p > 1$, and a weak $(1,1)$ estimate *uniformly in* t. Since the $H_t f$ converge for $f \in \mathcal{S}(\mathbb{R}^d)$ we see that they converge too for $f \in L^p(\mathbb{R})$ in the L^p-norm when $p > 1$ and in measure if $p = 1$. Both definitions of course agree because they agree on a dense class, that is

$$\lim_n H\varphi_n = \lim_n \lim_t H_t\varphi_n = \lim_t \lim_n H_t\varphi_n = \lim_t H_t f.$$

We claim now that in this later definition the $H_t f$ can be replaced by the $Q_t f$ and that moreover the limit exists a.e. pointwise for both. In case $p > 1$ this is easy for we have $Q_t\varphi_n = P_t H\varphi_n$, and therefore, since $P_t, Q_t \in L^{p'}(\mathbb{R})$, it follows that $Q_t f = P_t Hf$. Using that P_t is an approximation of the identity and theorem 1 we thus see that

$$Hf = \lim_t Q_t f, Hf = \lim_t H_t f,$$

boths limits being in the L^p-norm and also pointwise a.e.

For $f \in L^1(\mathbb{R}), Hf$ is the limit in measure of the $H_t f$. To see that the $Q_t f$ converge in measure to Hf too and that in both cases there is a.e.. convergence we cannot apply the same argument as before, because we cannot pass to the limit in $Q_t\varphi_n = P_t H\varphi_n$, as $P_t g$ does not make sense for $g \in L^{1,\infty}$.

To study this question we need to introduce the maximal functions

$$H^*f(x) = sup_t|H_tf(x)|, Q^*f(x) = \sup_t |Q_tf(x)|, f \in L^p(\mathbb{R}), 1 \le p < +\infty.$$

Theorem 7. • For $f \in L^p(\mathbb{R}), 1 < p < \infty$, one has

$$Q^*f(x) \le CM(Hf)(x), H^*f(x) \le C(M(Hf)(x) + Mf(x)),$$

for some constant C, where Mf denotes the Hardy-Littlewood maximal function of f.

- As a consequence, H^*, Q^* satisfy, a strong (p,p) estimate for $1 < p < +\infty$.
- H^*, Q^* satisfy a weak $(1,1)$- estimate.

Proof. The first inequality in the first part follows from $Q_tf = P_tHf$ and theorem 1. The first one then follows from, with the notations of theorem 3

$$|H_tf(x)| = |Q_tf(x)| + |H_tf(x) - Q_tf(x)| \le |Q_tf(x)| + |\Psi_tf(x)| = \\ |P_tHf(x)| + |\Psi_tf(x)| \le M(Hf)(x) + M(f)(x).$$

Given that M satisfies a strong (p,p) estimate and that H is bounded in L^p, the second part follows. For the third part we must slightly modify the proof of Kolmogorov's theorem. With the same notations, and using that we know that H^* satisfies a strong $(2,2)$ estimate, again everything is reduced to prove that

$$|\{x \notin E^* : |H^*b(x)| > \lambda\}| \le C\frac{\|f\|_1}{\lambda},$$

and again we need a pointwise estimate of $H^*b(x)$ for $x \notin E$. Fixed such x and one k, three cases can happen: either $(x-t, x+t)$ contains I_k, they are disjoint, or else one and only one of the two end points $x \pm t$ is in I_k. In the first case $H_tb_k(x) = 0$, in the second case $H_tb_k(x) = Hb_k(x)$ and then we have the estimate already proved

$$|H_tb_k(x)| \le C\frac{|I_k|}{|x-c_k|^2}\int_{I_k}|b_k(y)|\,dy.$$

In the third case, since $x \notin E^*$, one has that $I_k \subset (x-3t, x+3t), |x-y| > \frac{t}{3}, y \in I_k$, so that

$$|H_tb_k(x)| \le \int_{I_k}\frac{|b_k(y)|}{|x-y|}\,dy \le \frac{3}{t}\int_{x-3t}^{x+3t}|b_k(y)|\,dy.$$

Altogether we can write that in all three cases

$$|H_tb_k(x)| \le C\frac{|I_k|}{|x-c_k|^2}\int_{I_k}b_k(y)\,dy + \frac{3}{t}\int_{x-3t}^{x+3t}b_k(y)\,dy,$$

noticing that the last term, fixed x, can appear only twice. Adding on k, this shows

$$|H_t b|(x)| \leq \sum_k \frac{|I_k|}{|x-c_k|^2}\|b_k\|_1 + \frac{3}{t}\int_{x-3t}^{x+3t} b(y)\,dy.$$

Taking the supremum in t we get

$$H^* b(x) \leq \sum_k \frac{|I_k|}{|x-c_k|^2}\|b_k\|_1 + Mb(x).$$

The set where $Mb > \frac{\lambda}{2}$ has measure less than $C\frac{\|b\|_1}{\lambda} \leq C\frac{\|f\|_1}{\lambda}$ by the weak estimate satisfied by M, while the measure of the set where the above sum is bigger than $\lambda/2$ is treated exactly as in the proof of Kolmogorov's theorem. this shows that H^* satisfies a weak $(1,1)$ estimate, and since $Q^* f \leq H^* f + CMf$, the same holds for $Q^* f$. $\square$

The last statement implies of course that for $f \in L^1(\mathbb{R})$,

$$\lim_t H_t f(x) = Hf(x), \lim_t Q_t f(x) = Hf(x),$$

both in measure and pointwise a.e.

Of course, all these facts hold as well for the Cauchy transform, in particular the Plemej result holds a.e. for $f \in L^1(\mathbb{R})$.

The definition of H in $L^2(\mathbb{R})$ amounts to

$$Hf(x) = -i \lim_{R\to\infty} \int_{-R}^{R} sign(\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi,$$

the limit being taken in L^2-norm. Now, if $f \in L^p(\mathbb{R}), 1 \leq p < 2$, then $\hat{f} \in L^{p'}(\mathbb{R})$, the right hand side makes and it is then a natural question whether the above holds true with convergence in $L^p(\mathbb{R})$. this turns out to be true but we need postponing the proof.

On the other hand, it is easy to see that H cannot satisfy a strong $(1,1)$ estimate, that is, Hf is not integrable in general for $f \in L^1(\mathbb{R})$. For instance, a computation shows that the Hilbert transform of $1_{[0,1]}$ is $\frac{1}{\pi}\log(\frac{|x|}{|x-1|})$.

Notice too that if Hf is integrable then $\widehat{Hf}$ must be continuous and this is possible only if $\hat{f}(0) = 0$, that is, f has zero integral. Along this line, notice that if f is compactly supported then for x big enough

$$xHf(x) = \frac{1}{\pi}\int \frac{f(y)x}{x-y}\,dy,$$

has limit $\frac{1}{\pi}\int f$ as $x \to \infty$ and so it is not integrable if $\int f \neq 0$.

Let us study in some detail H on the Schwarz class. First, note that the above holds as well for $f \in \mathcal{S}(\mathbb{R}^d)$: namely we write for $|x|$ big

$$\pi Hf(x) = \int_{|x-y|<1} \frac{f(y)-f(x)}{x-y}\,dy + \int_{|x-y|>1} \frac{f(y)}{x-y}\,dy.$$

In the second term,

$$|x\frac{f(y)}{x-y}| \leq (1+|y|)|f(y)|.$$

In the second one, by Taylor expansion at y, $f(x)-f(y) = f'(y)(x-y)+R$, with $|R| \leq \frac{1}{2}|f''(z)||x-y|^2$ for some z between x and y; since $|z| > ||y|-1| > c|y|$ and f'' is rapidly decreasing we see that $R = O(|y|^{-2})$; altogether the first term is bounded by $|f'(y)|+|y|^{-2}$ uniformly in x. By dominated convergence we conclude that $\lim_{x\to\infty} xHf(x) = \frac{1}{\pi}\int f$. If $\int f = 0$, then we can substract $\frac{1}{x}$ from the kernel and get

$$Hf(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} f(y)(\frac{1}{x-y} - \frac{1}{x})\,dy = \frac{1}{x}H(yf)(x),$$

so that $x^2Hf(x)$ has limit $\frac{1}{\pi}\int yf$. Iterating we see that if all the moments $\int x^k f$ are zero, then $x^k Hf(x)$ has limit zero at ∞ for all k.

Now $D^{(n)}Hf = D^{(n)}(f * \frac{1}{\pi}p.v.\frac{1}{x}) = (D^{(n)}f) * \frac{1}{\pi}p.v.\frac{1}{x} = H(D^{(n)}f)$; on the other hand, integration by parts shows that all moments of $D^{(n)}f$ are zero if $n \geq 1$. It follows then that $x^k D^{(n)}Hf(x)$ has limit zero at infinity for all k and all $n \geq 1$. The only obstacle for Hf to be in $\mathcal{S}(\mathbb{R}^d)$ is $n = 0$ and this holds if and only if all moments of f are zero. Incidentally we can see this in a clear way on the Fourier transform side, because for $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$, the function $sign(\xi)\hat{f}(\xi)$ is in $\mathcal{S}(\mathbb{R}^d)$ if and only if $\hat{f}$ is flat at zero, that is, it has all derivatives zero at zero, which exactly says that the moments of f are zero.

3.8 Multipliers of $L^p(\mathbb{R}^d)$

We will see that the boundedness of the Hilbert transform has some important consequences. But first we explain some general facts about multipliers, and in doing so we place ourselves again in a general dimension $d \geq 1$.

We consider bounded linear operators $T : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d), 1 \leq p, q \leq \infty$ that commute with translations. A general theorem of Hormander (see Harmonic Analysis course) says that if p, q are both finite such an operator is given by convolution with a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d), Tf = f * u$ so that its action on $\varphi \in \mathcal{S}(\mathbb{R}^d)$ can be read $\widehat{T\varphi}(\xi) = \hat{\varphi}(\xi)m(\xi)$, with $m = \hat{u} \in \mathcal{S}'(\mathbb{R}^d)$.

We denote by $\mathcal{M}^{p,q}(\mathbb{R}^d)$ the space of all translation invariant operators from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$. A first point to remark is that we may assume that $p \leq q$:

Theorem 8. $\mathcal{M}^{p,q}(\mathbb{R}^d)$ reduces to zero if $p > q$.

Proof. The proof is similar to what was shown when seeing that the Fourier transform cannot be bounded from L^p to $L^{p'}$ if $p > 2$. Let us consider $f \in \mathcal{S}(\mathbb{R}^d)$ such that $Tf \neq 0$ and g a sum of N translates of $f, g(x) = \sum_{k=1}^N f(x-\lambda_k)$. If the λ_k are chosen spread enough we will have that $\|g\|_p = N^{\frac{1}{p}}\|f\|_p$. On the other hand, since T is translation invariant, $Tg(x) = \sum_{k=1}^N Tf(x-\lambda_k)$. But for an arbitrary non zero $h \in L^q(\mathbb{R}^d), q < \infty$,

$$(\int_{\mathbb{R}^d} |\sum_{k=1}^N h(x-\lambda_k)|^q \, dx,)^{\frac{1}{q}}$$

behaves like $N^{\frac{1}{q}}$ if the λ_k tend to infinity in a spread way (this is proved first for h with compact support). Thus we would have

$$N^{\frac{1}{q}} \leq C\|T\|N^{\frac{1}{p}},$$

for all N, and so $q \geq p$ if T is not zero. □

We remind that in two cases we have already characterized the space $\mathcal{M}^{p,q}(\mathbb{R}^d)$. Namely, when $p = q = 1, \mathcal{M}^{p,q}(\mathbb{R}^d)$ is the space of finite complex Borel measures, that is, convolution with a finite measure is the general translation invariant operator in $L^1(\mathbb{R}^d)$. Also, when $p = q = 2$, we know that the translation invariant operators in $L^2(\mathbb{R})$ correspond exactly with the bounded multipliers m.

The case $p = q = \infty$ is exceptional. Of course convolution with a finite measure is an example, but there are translation invariant operators in $L^\infty(\mathbb{R}^d)$ that are not given by convolution. For example, let S be defined by

$$Sf = \lim_{R\to+\infty} \frac{1}{R}\int_0^R f(x)\,dx,$$

on the space of bounded periodic functions, and extend it to $L^\infty(\mathbb{R})$ using Hahn-Banach Theorem. Then T is a continuous linear operator onto the space of constants functions that commutes with translations and is not given by convolution because its action on test functions is zero.

So from now on we assume that $p \leq q, q > 1$ and leave the case $p = q = \infty$ aside. For such T, its transpose T^t is the operator $T^t : L^{q'}(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$ defined by

$$\langle Tf, g\rangle = \int_{\mathbb{R}^d} Tfg dx = \langle f, T^t g\rangle = \int_{\mathbb{R}^d} fT^t g, f \in L^p(\mathbb{R}^d), g \in L^{q'}(\mathbb{R}^d).$$

Then T^t is translation invariant and $T \mapsto T^t$ establishes a bijection between $\mathcal{M}^{p,q}(\mathbb{R}^d)$ and $\mathcal{M}^{q',p'}(\mathbb{R}^d)$. In general, if T is given by a kernel K

$$Tf(x) = \int_{\mathbb{R}^d} f(y)K(x,y)\,dy,$$

then

$$T^t g(y) = \int_{\mathbb{R}^d} g(x)K(x,y)\,dx,$$

that is, T^t is given by the kernel $K^t(x,y) = K(y,x)$. When it is translation invariant then $K(x,y) = K(x-y)$. In terms of the multiplier, if $m(\xi)$ is the multiplier of T, then $m(-\xi)$ is the multiplier of T^t. Since obviously all L^p spaces are stable by the reflection operator, we see that in fact $\mathcal{M}^{p,q}(\mathbb{R}^d)$ and $\mathcal{M}^{q',p'}(\mathbb{R}^d)$ are equal. Besides, by the Riesz interpolation theorem, we know that then T will be bounded too from $L^r(\mathbb{R}^d)$ to $L^s(\mathbb{R}^d)$ if $r = r_t, s = s_t$ are the required convex combinations. When $p = q$, we call $\mathcal{M}^p(\mathbb{R}^d)$ the space of all translation invariant operators from $L^p(\mathbb{R}^d)$ to itself, $1 < p < \infty$. Thus $\mathcal{M}^{p'}(\mathbb{R}^d) = \mathcal{M}^p(\mathbb{R}^d)$ and we can assume $1 < p \leq 2$. By the Riesz interpolation theorem just mentioned, it follows that for $1 \leq p < q \leq 2$,

$$\mathcal{M}^p(\mathbb{R}^d)\mathcal{M}^q(\mathbb{R}^d)\mathcal{M}^2(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$$

.

Thus all multipliers m of $L^p(\mathbb{R}^d)$ are also multipliers of $L^2(\mathbb{R}^d)$ and so they are bounded. For $p = 1, \mathcal{M}^1(\mathbb{R}^d)$, is the smallest, it consists in convolution with measures μ, that is bounded in L^p, and has multiplier $m = \widehat{\mu}$, a bounded function

3.9 Some multipliers of $L^p(\mathbb{R})$

We have seen that H is a multiplier of $L^p(\mathbb{R}), 1 < p < \infty$, and the same holds for the Cauchy transform, the projection onto the holomorphic subspace. From this it is immediate to see that any filter $S_{a,b}$ in frequency, that is, the operator with multiplier

$$m(\xi) = 1_{[a,b]}(\xi),$$

is also an L^p multiplier. If M_a denotes the operation of multiplication with $e^{2\pi i a x}$, which is obviously bounded and that corresponds to the translation τ_a in frequency, just notice that

$$S_{a,b} = \frac{i}{2}(M_a H M_{-a} - M_b H M_{-b}).$$

Moreover we get from this that their operator norm is uniformly bounded in a, b,

$$\|S_{a,b} f\|_p \leq C_p \|f\|_p.$$

Let us consider $S_R = S_{-R,R}$ which for $f \in L^2(\mathbb{R})$ is given by

$$S_R f(x) = \int_{-R}^{R} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi.$$

We know that $S_R f \to f$ in $L^2(\mathbb{R})$ if $f \in L^2(\mathbb{R})$ by the very definition of the Fourier transform in L^2. If we can exhibit a dense subspace of $L^p(\mathbb{R})$ in which $S_R f \to f$ then by the uniform boundedness principle this would hold for all f. That space is simply the subspace of $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\hat{\varphi}$ compactly supported. Obviously $S_R\varphi = \varphi$ for R big enough. To see that it is dense it suffices to approximate $f \in \mathcal{S}(\mathbb{R}^d)$ in L^p-norm by such functions. If Ψ is C^∞ and has compact support and equals 1 at 0, then $\hat{\Psi}_t(x) = t^{-d}\hat{\Psi}(x/t)$ is an approximation of the identity so $f * \hat{\Psi}_t \to f$ in L^p, and has Fourier transform equal to $\hat{f}(\xi)\Psi(t\xi)$, with compact support.

We have thus shown that $S_R f \to f$ in $L^p(\mathbb{R}), 1 < p < \infty$. An analogous argument would show that for $p = 1$ the convergence is in measure. For $1 \le p \le 2, \hat{f} \in L^{p'}(R)$ and $S_R f$ is given by the above expression, but this is not the case for $p > 2$. Along the same lines, applying H to $f = \lim_R S_R f$ we see that

$$Hf(x) = -i \lim_{R\to\infty} \int_{-R}^{R} sign(\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi} d\xi, f \in L^p(\mathbb{R}), 1 < p \le 2.$$

Theorem 9. Assume that m is a function of bounded variation in $\mathbb{R}$, that is

$$\sup \sum_{j=1}^{N} |m(x_j) - m(x_{j-1})| \le C < +\infty,$$

for all points $x_0 < x_1 < \cdots < x_N$. Then m is a multiplier for $L^p(\mathbb{R}^d), 1 \le p < +\infty$.

Proof. We may assume that m is up to an additive constant the distribution function of a finite measure,

$$m(\xi) = c + \int_{-\infty}^{\xi} d\mu(t),$$

that we may write in a compact form

$$m = c + \int_{-\infty}^{+\infty} 1(t, +\infty) d\mu(t).$$

Then the operator T_m with symbol m is

$$T_m = cI + \int_{-\infty}^{+\infty} S_{(t,+\infty)} d\mu(t),$$

that is, an infinite linear combination with summable coefficients of the $S_{(t,+\infty)}$ which are uniformly bounded in L^p by C_p, and so T_m is bounded in L^p. □

It is easy to obtain examples of multipliers in general dimension from multipliers in $d = 1$. Indeed, if $m(\xi_1)$ is a multiplier in $L^p(\mathbb{R})$, then this same function viewed as independent from $\xi_2, \cdots, \xi_d$ is a multiplier in $L^p(\mathbb{R}^d)$, with the same norm, by Fubini's theorem. On the other hand, if m is a multiplier in $L^p(\mathbb{R}^d)$, then $m(\xi+a), m(\lambda\xi), m(A\xi)$ with $a \in \mathbb{R}^d, \lambda > 0$, and A an unitary matrix are also multiplier with the same norm. Combining both things, we see that the characteristic function of a half-space is a multiplier for $L^p(\mathbb{R}^d), 1 < p < \infty$. The characteristic function of a convex polyhedra with N faces is the product of the N characteristic functions of half-spaces, and so it is a multiplier. This implies, in a similar way as we saw before

Theorem 10. The characteristic function of a convex polyhedra U is a multiplier of $L^p(\mathbb{R}^d), 1 < p < \infty$. In particular, $\lim_{\lambda\to\infty} S_{\lambda U} f = f$ in L^p.

The situation is quite different for other convex bodies. For instance, a celebrated theorem of C. Fefferman establishes that the characteristic function of a ball is a multiplier only for $p = 2$.