Calderon Zygmund singular integral operators

An analysis of the proof of the weak (1,1) estimate for the Hilbert transform reveals that it is based, first, in the Calderon-Zygmund decomposition, and secondly in the fact that $Hf(x)$, for f compactly supported and $x \notin \operatorname{spt} f$ has a nice expression

$$Hf(x) = \frac{1}{\pi} \int \frac{f(y)}{x-y} \, dy$$

that in case f has mean zero goes to

$$Hf(x) = \frac{1}{\pi} \int f(y) \left\{ \frac{1}{x-y} - \frac{1}{x-\bar{y}} \right\} dy$$

with $\bar{y}$ suitably chosen. Of course, we used too the (2,2) strong-estimate.

Let us define precisely a CZ operator. The more general setting is the one already encountered with the Hardy-Littlewood maximal function. We have balls $B(x,\delta)$ defined by a pseudometric (symmetric) $\rho(x,y)$ (with the engulfing property) and a doubling measure μ. We call this setting an homogeneous space setting

Definition. In the above context, a Calderon-Zygmund operator is a linear operator T such that

(a) T is bounded in $L^q(\mu)$, $\|Tf\|_q \le A\|f\|_q$, $1<q<\infty$

(b) There exists a kernel $K(x,y)$ defined for $x \neq y$ s.t. if $f \in L^q$ has compact support

$$Tf(x) = \int f(y)\, K(x,y)\, d\mu(y), \quad x \notin \operatorname{spt} f$$

(c) The kernel $K(x,y)$ satisfies: for every ball $B(y,\delta)$ and $\bar{y} \in B(y,\delta)$, for some $c>1$,

$$(*) \quad \int_{{}^cB(\bar{y},c\delta)} |K(x,y) - K(x,\bar{y})|\, d\mu(x) \le A$$

(that means we integrate over $\rho(x,\bar{y}) \ge c\delta$, and $\rho(y,\bar{y}) \le \delta$, $c>1$)

We will prove, analyzing the proof for H, that

$\boxed{\text{Theorem}}$ A CZ-operator is weak $(1,1)$, and hence strong (p,p) for $1<p\le q$

If the adjoint operator with kernel $K^t(x,y) = K(y,x)$ satisfies

$$\int_{\rho(x,\bar{y}) \ge c\delta} |K^t(x,y) - K^t(x,\bar{y})|\, d\mu(x) \le A, \quad \rho(y,\bar{y}) \le \delta$$

that is

$$\int_{\rho(x,\bar{y})\geq c\delta} |K(y,x)-K(\bar{y},x)|\, d\mu(x) \leq A$$

or

$$\int_{\rho(\bar{x},y)\geq c\delta} |K(x,y)-K(\bar{x},y)|\, d\mu(y) \leq A \qquad \rho(x,\bar{x})\leq\delta$$

then T will be bounded for $q'\leq p<+\infty$. If $q=2$, (the case of the Hilbert transform), it will be strong (p,p) for all p, $1<p<+\infty$.

If K is symmetric ($K=K^t$), then $q=2$, and is (p,p) $\forall p$, $1<p<\infty$

A very general condition that implies $(*)$ in the more general setting is as follows. Define

$$V(x,y) = \inf\{\mu(B(y,\delta)) : x\in B(y,\delta)\}$$

the volume of the smallest ball $B(y,\delta)$ containing x (note that $V(x,y)\sim V(y,x)$ and in the euclidean setting $V(x,y)=c_d|x-y|^{+d}$)

<u>Prop</u> Assume that for some modulus η of Dini type, that is $\int_0^1 \frac{\eta(s)}{s}\,ds<+\infty$

$$|K(x,y)-K(x,\bar{y})| \leq \eta\left(\frac{\rho(y,\bar{y})}{\rho(x,\bar{y})}\right)\frac{1}{V(x,\bar{y})}$$

for $\rho(x,\bar{y})\geq c\rho(y,\bar{y})$. Then $(*)$ holds

<u>Proof</u>. We split the integral over $\rho(x,\bar{y}) \geq c\delta$, $\delta = \rho(y,\bar{y})$, as

$$\sum_{k=0}^{\infty} \int_{B(\bar{y},2^{k+1}c\delta)\setminus B(\bar{y},2^{k}c\delta)} |K(x,y)-K(x,\bar{y})|\,d\mu(x)$$

If $x \notin B(\bar{y},2^{k}c\delta)$, $\rho(x,\bar{y}) \geq 2^{k}c\delta$, and $V(x,\bar{y}) \geq \mu(B(\bar{y},2^{k}c\delta))$, and hence

$$\leq \sum_{k=0}^{\infty} \mu(B(\bar{y},2^{k+1}c\delta))\,\eta\left(\frac{1}{2^{k}c}\right)\mu(B(\bar{y},2^{k}c\delta))^{-1}$$

which by the doubling condition is bounded by

$$\text{const} \sum_{k=0}^{\infty} \eta\left(\frac{1}{2^{k}c}\right) \leq \text{const} \int_0^1 \frac{\eta(s)}{s}\,ds < +\infty.$$

The most usual form where the condition in the Prop appears (in Euclidean balls and μ = Lebesgue measure) is

$$|K(x,y)-K(x,\bar{y})| \leq A\frac{|y-\bar{y}|^{s}}{|x-\bar{y}|^{d+s}} \qquad |x-\bar{y}| \geq 2|y-\bar{y}|$$

This is automatically satisfied if K is smooth away from $\Delta = \{x=y\}$ and satisfies

$$|\nabla_y K(x,y)| \leq \frac{C}{|x-y|^{d+1}}$$

This is easily seen just using the mean-value theorem: if $\varphi(t) = K(x, ty + (1-t)\bar{y})$ then

$$|K(x,y) - K(x,\bar{y})| = |\varphi(1) - \varphi(0)| \leq \int_0^1 |\varphi'(t)|\,dt \leq$$

$$\leq |\bar{y} - y| \int_0^1 |\nabla_y K(x, ty + (1-t)\bar{y})|\,dy \leq$$

$$\leq \frac{|\bar{y}-y|}{|x-\bar{y}|^{d+1}}$$

because $|x - (ty + (1-t)\bar{y})| \sim |x-\bar{y}|$ for $2|y-\bar{y}| \leq |x-\bar{y}|$. //

As in the case for H, the main tool in proving the Theorem is the Calderon-Zygmund decomposition of function in L^1, that we need in an homogeneous space setting.

<u>But</u> before let us see some subjects where Calderon-Zygmund operators appear naturally.

In many situations, the L^q-estimate in the hypothesis (usually $q=2$) comes from a different understanding of T (through Fourier transform in case it is translation-invariant, or through integration by parts). The following examples show this

SOME EXAMPLES WHERE CZ-OPERATORS ARISE

(A) The Laplacian $\Delta\varphi$ of $\varphi \in C_c^2(\mathbb{R}^d)$ controls all derivatives of order 2

By this we mean that

$$\int_{\mathbb{R}^d} \left(\frac{\partial^2\varphi}{\partial x_k \partial x_j}\right)^2 dx \le C \int_{\mathbb{R}^d} |\Delta\varphi|^2 dx$$

This ~~trivially~~ holds with $C = 1$ by integration by parts.

$$\int \frac{\partial^2\varphi}{\partial x_k \partial x_j} \frac{\partial^2\varphi}{\partial x_k \partial x_j} = -\int \frac{\partial\varphi}{\partial x_j} \frac{\partial^3\varphi}{\partial x_k^2 \partial x_j} =$$

$$= \int \frac{\partial^2\varphi}{\partial x_j^2} \frac{\partial^2\varphi}{\partial x_k^2} \le \|D_j^2\varphi\|_2 \|D_k^2\varphi\|_2 \le \|\Delta\varphi\|_2^2$$

What about control in other norms, for instance $L^p(\mathbb{R}^d)$? Is it true that

$$\int_{\mathbb{R}^d} \left|\frac{\partial^2\varphi}{\partial x_k \partial x_j}\right|^p dx \le C_p \int_{\mathbb{R}^d} |\Delta\varphi|^p dx$$

Let us ask ourselves: what is the operator $T : L^2(\mathbb{R}^d) \to L^2$ that sends $\Delta\varphi$ to $\dfrac{\partial^2\varphi}{\partial x_k \partial x_j}$?

Since everything is invariant by translations we use Fourier transforms and find the answer easily. Just note

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^d} \varphi(x)\, e^{-2\pi i \xi \cdot x}\, dx \;;\; \varphi(x) = \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\, e^{-2\pi i \xi\cdot x}\, d\xi$$

$$D_{jk}\varphi = \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \longleftrightarrow (-2\pi i\xi_j)(-2\pi i \xi_k)\, \hat{\varphi}(\xi) = -4\pi^2 \xi_k \xi_j\, \hat{\varphi}(\xi)$$

$$\Delta\varphi \longleftrightarrow -4\pi^2 |\xi|^2 \hat{\varphi}(\xi)$$

Hence $\widehat{T_m f}(\xi) = m(\xi)\hat{f}(\xi)$ with $m(\xi) = \dfrac{\xi_k \xi_j}{|\xi|^2}$ is the bounded operator on $L^2(\mathbb{R}^d)$ that maps $\Delta\varphi$ to $D_{jk}\varphi$

Which is its expression as convolution ?

We must compute the tempered distribution K such that

$$\hat{K} = \frac{\xi_k \xi_j}{|\xi|^2}$$

(a) We start from the tempered distribution G such that $-4\pi|\xi|^2 \hat{G} = 1$, the fundamental solution of Δ (this is not the same as saying that $\hat{G} = -1/4\pi|\xi|^2$ if $d \le 2$, for in this case $1/|\xi|^2$ is not a tempered distribution).

This fundamental solution G, as it is well know, is

$$G(x) = \begin{cases} \dfrac{1}{(2-d)}\dfrac{1}{c_d}|x|^{2-d} & \text{if } d>2, \quad c_d = \sigma(S^{d-1}) \\ \dfrac{1}{2\pi}\log|x| & d=2 \\ \dfrac{1}{2}|x|, & d=1. \end{cases}$$

We will in fact prove it is a fund. solution

(b) Derivatives as distributions of homogeneous functions

Suppose f is of class C^1 away from zero and homogeneous of degree $-\alpha$, that is $f(\lambda x) = \lambda^{-\alpha} f(x)$, $\lambda > 0$. If $\alpha < d$ then $f \in L^1_{loc}(\mathbb{R}^d)$ and it makes sense to compute its derivatives in the distribution sense.

$$\langle D_k f, \varphi \rangle = -\langle f, D_k \varphi \rangle, \quad \varphi \in C_c^\infty$$

$$= -\int_{\mathbb{R}^d} f(x)\, D_k \varphi(x)\, dx = -\lim_{\varepsilon} \int_{|x| \geq \varepsilon} f(x)\, D_k \varphi(x)\, dx$$

Now we have by the divergence theorem, for all $\psi \in C^1(\mathbb{R}^d \setminus \{0\})$

$$\int_{|x| = \varepsilon} \psi(x) \frac{x_k}{|x|}\, dS = -\int_{|x| \geq \varepsilon} \frac{\partial \psi}{\partial x_k}\, dx$$

Applying this to $\psi = f \varphi$

$$\int_{|x| = \varepsilon} f(x)\varphi(x) \frac{x_k}{|x|}\, dS = -\int_{|x| \geq \varepsilon} \left[(D_k f)\varphi + f(D_k \varphi) \right] dx$$

$$\langle D_k f, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} (D_k f)(x)\, \varphi(x)\, dx + \lim_{\varepsilon \to 0} \int_{|x| = \varepsilon} f(x)\varphi(x) \frac{x_k}{|x|}\, dS$$

Introduce polar coordinates: $x = r\omega$, $|\omega| = 1$, $dx = r^{d-1}\,dr\,d\sigma(\omega)$. The first limit is

$$I = \lim_{\varepsilon} \int_{\varepsilon}^{\infty} \int_{|\omega|=1} r^{-\alpha-1} D_k f(\omega)\, \varphi(r\omega)\, r^{d-1}\, d\sigma(\omega)$$

(because $D_k f$ is homogeneous of degree $-\alpha-1$)

$$I = \lim_{\varepsilon} \int_{\varepsilon}^{\infty} r^{-\alpha+d-2} \left(\int_{|\omega|=1} D_k f(\omega)\, \varphi(r\omega)\, d\sigma(\omega) \right) dr .$$

The second limit is

$$II = \lim_{\varepsilon} \varepsilon^{-\alpha+d-1} \int_{|\omega|=1} f(\omega)\, \varphi(\varepsilon\omega)\, \omega_k \, d\sigma(\omega)$$

Thus we see that if $\alpha < d-1$ the first limit exists (that is $D_k f \in L^1_{loc}$) and the second is zero.

$$D_k f \text{ (as a distribution)} = D_k f \text{ (as a function)}.$$

If $\alpha = d-1$ the second limit is

$$\varphi(0) \int_{|\omega|=1} f(\omega)\, \omega_k \, d\sigma(\omega)$$

and we proceed to study more carefully the first

$$I = \lim \int_{\varepsilon}^{\infty} \frac{1}{r} \left(\int_{|w|=1} D_k f(w)\, \varphi(rw)\, d\sigma(w) \right) dr$$

Lemma If f is homogeneous of degree $1-d$ then

$$\int_{|w|=1} D_k f(w)\, d\sigma(w) = 0 .$$

Proof. Take $\rho(r)$ equal to 0 for $0 \le r \le 1$ and equal to 1 for $r \ge 2$, $\rho \in C_c^{\infty}(\mathbb{R})$. Then

$$0 = \int_{\mathbb{R}^d} D_k \left[\rho(|x|) f(x) \right] dx = \int_{\mathbb{R}^d} \rho'(|x|) \frac{x_k}{|x|} f(x)\, dx +$$

$$+ \int_{\mathbb{R}^d} \rho(|x|) D_k f(x)\, dx \overset{\text{def}}{=} A + B$$

$$A = \int_0^{\infty} \int_{|w|=1} \rho'(r)\, w_k\, f(rw)\, r^{d-1}\, dr\, d\sigma(w) = \left(\int_0^{\infty} \rho'(r)\, dr \right) \left(\int_{|w|=1} w_k f(w)\, d\sigma(w) \right) = 0.$$

$$B = \int_0^{\infty} \int_{|w|=1} \rho(r)\, D_k f(rw)\, r^{d-1}\, dr\, d\sigma(w) = \left(\int_0^{\infty} \frac{\rho(r)}{r}\, dr \right) \left(\int_{|w|=1} D_k f(rw)\, d\sigma(w) \right)$$

Then

$$I = \lim_{\varepsilon} \int_{\varepsilon}^{\infty} \frac{1}{r} \left(\int_{|w|=1} (D_k f(w)) \left[\varphi(rw) - \varphi(0) \right] d\sigma(w) \right) dr$$

exists because $\varphi(rw) - \varphi(0) = O(r)$ and the singularity is cancelled. The above is the same as

$$\lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} D_k f(x) \left(\varphi(x) - \varphi(0) \right) dx$$

and we call it $\boxed{\text{v.p. } D_k f}$. Thus we have

<u>Theorem</u>. If $\alpha < d-1$, $D_k f = D_k f$ (strictly $D_k T_f = T_{D_k f}$)

If $\alpha = d-1$

$$D_k T_f = \text{p.v. } D_k f + c\, \delta_0$$

with $c = \int_{|w|=1} f(w)\, w_k \, d\sigma(w)$

c) We compute now the kernel $K_{jk} = D_{jk} G$

Assume first $d>2$, then $G(x) = \frac{1}{(2-d)c_d} |x|^{2-d}$

is homogeneous of degree $2-d$ ($\alpha = d-2$). The first derivatives are still integrable hence

$$D_j G(x) = \frac{1}{c_d} \frac{x_j}{|x|^d}$$

Now $D_j G$ is homogeneous of degree $1-d$ ($\alpha = d-1$), so

$$D_j D_k G = \text{p.v. } D_j D_k G + c_{jk} \delta_0$$

with $c_{jk} = \int_{|w|=1} w_k D_j G(w)\, d\sigma(w) = \frac{1}{c_d} \int_{|w|=1} w_k w_j \, d\sigma(w)$

Note that $c_{jk} = 0$ if $j \neq k$ and $c_{jj} = \frac{1}{d}$.

$$D_j D_k G = \frac{1}{c_d} \left\{ \frac{\delta_{jk}}{|x|^d} - d \frac{x_j x_k}{|x|^{d+2}} \right\} + \frac{\delta_{jk}}{d} \delta_0$$

p.v.

Notice that indeed

$$\sum_{j=1}^{d} D_j D_j G = \delta_0$$

The computation if $d=2$ is a bit different. In the first step it works the same because the first derivatives of $\log|x|$ are still integrable. So

$$D_j\left(\frac{1}{2\pi}\log|x|\right)=\frac{1}{2\pi}\frac{x_j}{|x|^2}$$

This is now homogeneous of degree -1 and it works exactly as before to ~~fi~~ compute T

$$D_{kj}\left(\frac{1}{2\pi}\log|x|\right)=\mathrm{pv}\frac{1}{2\pi}\left\{\frac{\delta_{jk}}{|x|^2}-2\frac{x_j x_k}{|x|^4}\right\}+\frac{\delta_{jk}}{2}\delta_0$$

(For $d=1$, $\frac{1}{2}|x|$ has derivative $\frac{1}{2}(\mathbb{1}_{(0,\infty)}-\mathbb{1}_{(-\infty,0)})$ and this has derivative δ_0).

In our situation we are of course interested in case $j\neq k$ and find that

$$K(x,y)=-\frac{d}{c_d}\frac{(x_j-y_j)(x_k-y_k)}{|x-y|^{d+2}}$$

Indeed

$$|D_y K(x,y)|\leq\frac{\mathrm{const}}{|x-y|^{d+1}}$$

and hence convolution with p.v. $D_{kj}\, G(x-y)$ is a bounded operator in $L^p(\mathbb{R}^d)$, $1<p<+\infty$. This shows

<u>Theorem</u>. For $1<p<+\infty$, there exists C_p such that

$$\| D_{kj}\, \varphi \|_p \le C_p \| \Delta \varphi \|_p \,, \quad \varphi \in C_c^\infty(\mathbb{R}^d)$$

More generally, if $\psi \in C_c^\infty(\mathbb{R}^d)$ the potential

$$G\psi(x) = \int_{\mathbb{R}^d} \psi(y)\, G(x-y)\, dy = (\psi * G)(x)$$

(that satisfies $\Delta(G\psi) = \psi$ in the sense of distributions), has second derivatives in $L^2(\mathbb{R}^d)$ ~~with~~ in the sense of distributions and

$$\| D_{jk}\, (G\psi) \|_2 \le C \|\psi\|_2$$

<u>Remark 1</u>. The functions $G\psi$, $\nabla(G\psi)$ are not in $L^2(\mathbb{R}^d)$.
As a consequence of the estimate

$$\| f * g \|_q \le \|f\|_{p,\infty} \|g\|_r \qquad 1<p,q,r<\infty$$

$\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$ we have $G\psi$, $\nabla(G\psi)$ in certain L^q

$(d>2)$

<u>Remark 2</u>. For $G\psi$ to be C^2 and satisfy $\Delta(G\psi) = \psi$ in the classical sense the condition

$$\psi \in \mathrm{Lip}_\varepsilon \quad \text{suffices}$$

2nd example The Hodge-Helmholtz decomposition of vector fields

Reminder about H-H decomposition of smooth vector fields

U domain in $\mathbb{R}^3$. We consider vector fields $\vec{X} \in L^2_{\mathbb{R}}(U)^3$ with inner product

$$\langle \vec{X}, \vec{Y} \rangle = \int_U \Big(\sum_{j=1}^{3} X_j(x) Y_j(x) \Big) dx$$

- If $\vec{X} \in C^1(U)$, $\vec{Y} = \vec{\nabla} \Phi$ with $\Phi \in C_c^\infty(U)$ then

$$\langle \vec{X}, \vec{\nabla} \Phi \rangle = - \langle \operatorname{div} \vec{X}, \Phi \rangle$$

That's why we define

__Dfn__ If $\vec{X} \in L^1_{loc}(U)^3$ and $h \in L^1_{loc}(U)$ we say that $\operatorname{div} \vec{X} = h$ in the weak sense if

$$\langle \vec{X}, \vec{\nabla} \Phi \rangle = - \langle h, \Phi \rangle = - \int h \phi, \quad \forall \Phi \in C_c^\infty$$

- Similarly, if $\vec{X} \in C^1(U)$, $\vec{Y} = \operatorname{curl} \vec{Z}$, $\vec{Z} \in C_c^\infty(U)^3$

$$\langle \vec{X}, \operatorname{rot} \vec{Z} \rangle = - \langle \operatorname{rot} \vec{X}, \vec{Z} \rangle$$

__Dfn__ If $\vec{X}, \vec{Y} \in L^1_{loc}(U)^3$ say $\operatorname{rot} \vec{X} = \vec{Y}$ in the weak sense if $\langle \vec{X}, \operatorname{rot} \vec{Z} \rangle = - \langle \vec{Y}, \vec{Z} \rangle$ $\forall \vec{Z} \in C_c^\infty(U)^3$

- A continuous $\vec{X}$ is locally conservative (it has a potential function, $\vec{X} = \vec{\nabla} h$) iff $\mathrm{rot}\, \vec{X} = 0$ in the weak sense
  If U is simply connected : locally $\rightarrow$ globally

- A continuous $\vec{X}$ is locally solenoidal (it has a potential vector $\vec{Y}$, $\vec{X} = \mathrm{rot}\, \vec{Y}$) iff $\mathrm{div}\, \vec{X} = 0$ in the weak sense
  If U has no holes : locally $\rightarrow$ globally

- Assume U is a nice domain, simply connected, no holes, with smooth boundary ∂U oriented by the exterior normal $\vec{N}$

  Fact: if $\vec{X}$ continuous is both conservative and solenoidal then there is φ harmonic on U such that $\vec{X} = \nabla h$, in particular $\vec{X}$ is C^∞. If $\langle \vec{X}, \vec{N} \rangle$ on ∂U then $\vec{X} \equiv 0$.

$\vec{X} = \nabla \varphi$, $\mathrm{div}\, (\nabla \varphi) = 0$ in the weak sense

means $\Delta \varphi = 0$ in the weak sense

$\langle \vec{X}, \vec{N} \rangle = \dfrac{\partial \varphi}{\partial N}$. //

$$A = \{ \vec{X} \in L^2(U)^3 : \mathrm{rot}\, \vec{X} = 0 \text{ in the weak sense} \}$$

$$B = \{ \vec{X} \in L^2(U)^3 : \mathrm{div}\, \vec{X} = 0 \quad \text{''} \quad \}.$$

closed subspaces of $L^2(U)^3$.

$A^\perp \subset B$

$B^\perp \subset A$

H-H decomposition

$\vec{X}$ smooth enough on U. Want to split

$$\vec{X} = \vec{X_1} + \vec{X_2}, \quad \vec{X_1} \in A, \vec{X_2} \in B$$

$$\operatorname{div} \vec{X} = \operatorname{div} \vec{X_1} + \operatorname{div} \vec{X_2} = \operatorname{div} \vec{X_1} \quad \Big| \rightarrow \Delta\psi = \operatorname{div} \vec{X}$$

Search ψ, $X_1 = \nabla\psi$

- We may impose a Neumann type condition, e.g. $\frac{\partial\psi}{\partial N} = \langle \vec{X}, \vec{N} \rangle$

So that $\langle \vec{X_1}, \vec{N} \rangle = \langle \nabla\psi, N \rangle = \frac{\partial\psi}{\partial N} = \langle \vec{X}, \vec{N} \rangle$

Therefore : in any possible decomposition, $\vec{X_1}$ is ~~known~~ ~~as~~ unique if $\langle \vec{X_1}, \vec{N} \rangle = \langle \vec{X}, \vec{N} \rangle$. Then X_2 satisfies $\operatorname{div} X_2 = 0$, $\langle X_2, N \rangle = 0$.

The decomposition is orthogonal and unique

$$\langle X_1, X_2 \rangle = \int (\vec{\nabla}\psi(x) \cdot X_2)$$

$$(\vec{\nabla}\psi \cdot X_2) = \operatorname{div}(\psi X_2) - \psi(\operatorname{div} X_2) = \operatorname{div}(\psi X_2)$$

$$\rightarrow \langle X_1, X_2 \rangle = \int_U \operatorname{div}(\psi \vec{X_2}) = \int_{\partial U} \psi \langle X_2, N \rangle dA = 0.$$

- We may impose other boundary conditions and get other decompositions

The decomposition on the whole of $\mathbb{R}^3$

Thm $\vec{X} \in C_c^\infty(\mathbb{R}^3)^3$,

$\Rightarrow \vec{X} = \vec{X_1} + \vec{X_2}$, $\vec{X_1} \in A$, $\vec{X_2} \in B$. Moreover, is

→ Unique with $\vec{X_1}, \vec{X_2}$ continuous, $\vec{X_i}(x) = O(|x|^{-\varepsilon})$

→ Orthogonal

Proof. Uniqueness: $\vec{X_1} + \vec{X_2} = \vec{Y_1} + \vec{Y_2} \Rightarrow$

$X_1 - Y_1 = Y_2 - X_2 = \nabla u$ harmonic vanishing at ∞

$\Rightarrow \nabla u = 0$

Orthogonal: as before

$$\langle X_1, X_2 \rangle = \lim_R \int_{|x| \le R} \operatorname{div}(\psi X_2) = \lim_R \int \psi \langle X_2, N \rangle dA$$

Solve $\Delta \psi = \operatorname{div} \vec{X}$ with fundamental solution

$$\psi(x) = \int \operatorname{div} \vec{X}(y)\, G(x-y)\, dy \sim |x|^{-1}$$

$$X_1(x) = \nabla \psi = \int \operatorname{div} \vec{X}(y)\, \nabla_x G(x-y)\, dy =$$

$$\boxed{= \frac{1}{4\pi} \int_{\mathbb{R}^3} \operatorname{div} \vec{X}(y) \frac{x-y}{|x-y|^3}\, dy} \sim |x|^{-2}$$

$\psi X_2 \sim |x|^{-3} \Rightarrow \langle X_1, X_2 \rangle = 0.$

Analogously with $\vec{X}_2$: search it $\vec{X}_2 = \mathrm{rot}\,\vec{Y}$

$\vec{Y}$ not unique ; but it is unique if we normalize it requiring $\vec{Y}$ solenoidal $(\mathrm{div}\,\vec{Y} = 0)$ $\vec{Y}(\infty) = 0$

(for if $\mathrm{rot}\,\vec{Y}_1 = \mathrm{rot}\,\vec{Y}_2$, $\mathrm{div}\,\vec{Y}_1 = \mathrm{div}\,\vec{Y}_2 = 0$, $\vec{Y}_i(\infty) = 0$
then $\vec{Y}_1 - \vec{Y}_2 = \nabla u$, u harmonic , $\nabla u(\infty) = 0 \Rightarrow \nabla u = 0$

Look for $\vec{Y}$, $\mathrm{div}\,\vec{Y} = 0$, $\vec{X}_2 = \mathrm{rot}\,\vec{Y}$

$$\mathrm{rot}\,\vec{X} = \mathrm{rot}\,\vec{X}_1 + \mathrm{rot}\,\vec{X}_2 = \mathrm{rot}\,\vec{X}_2 = \mathrm{rot}\,\mathrm{rot}\,\vec{Y} =$$

$$= \vec{\nabla}(\mathrm{div}\,Y) - \Delta\vec{Y} = -\Delta\vec{Y}$$

$$\vec{Y}(x) = -\int_{\mathbb{R}^3} \mathrm{rot}\,\vec{X}(y)\, G(x-y)\, dy$$

$$\boxed{\vec{X}_2 = \mathrm{rot}\,\vec{Y}(x) = \frac{1}{4\pi}\int \frac{y-x}{|y-x|^3} \wedge \mathrm{rot}\,\vec{X}(y)\, dy}$$

In the expression of φ, $\vec{Y}$ we can still integrate by parts in order to have them in terms of $\vec{X}$

$$\varphi(x) = -\int \vec{X}(y) \cdot \nabla_y G(x-y)\, dy =$$

$$= \frac{1}{4\pi}\int \vec{X}(y) \cdot \frac{x-y}{|x-y|^3}\, dy \qquad \text{(Biot-Savart)}$$

$$\vec{Y}(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y-x}{|x-y|^3} \wedge \vec{X}(y)\, dy$$

With these we find explicitly X_1, X_2 in terms of X, taking into account what was done before

$$\boxed{\begin{aligned} \vec{X}_1(x) &= \vec{\nabla}\varphi(x) = \text{p.v.}\ \frac{1}{4\pi}\int \underbrace{\nabla_x \frac{x-y}{|x-y|^3}}_{\text{matrix } \frac{\partial}{\partial x_j}\frac{x_i-y_i}{|x-y|^3}} \cdot \vec{X}(y)\,dy + \frac{1}{3}\vec{X}(x) \\ \vec{X}_2(x) &= \text{p.v.}\ \frac{1}{4\pi}\int \operatorname{rot}_x \frac{y-x}{|x-y|^3} \wedge \vec{X}(y) \end{aligned}}$$

<u>Theorem</u> Set $P(\vec{X}) = \vec{X}_1$, $Q(\vec{X}) = \vec{X}_2$ for $\vec{X} \in C_c^\infty(\mathbb{R}^3)^3$. Then $X = P(X) + Q(X)$ is the orthogonal decomposition, so that

$$\|P(X)\|_2 \leq \|X\|_2, \quad \|Q(X)\|_2 \leq \|X\|_2$$

$$\|X\|_2^2 = \|P(X)\|_2^2 + \|Q(X)\|_2^2$$

P, Q extend by continuity to $L^2(\mathbb{R}^3)^3$, and are the orthogonal projections on A, B, so that

$$L^2(\mathbb{R}^3)^3 = A \perp B$$

There are simple expressions in terms of the Fourier transform

$$\vec{X}(x) = \int \hat{X}(\xi)\, e^{2\pi i \xi x}\, d\xi$$

Decompose

$$\hat{X}(\xi) = \left(\hat{X}(\xi) \cdot \frac{\xi}{|\xi|}\right)\frac{\xi}{|\xi|} + \left(\frac{\xi}{|\xi|} \wedge \hat{X}(\xi)\right) \wedge \frac{\xi}{|\xi|}$$

Then

$$\vec{X}_1(x) = \int \left(\hat{X}(\xi) \cdot \frac{\xi}{|\xi|}\right)\frac{\xi}{|\xi|}\, e^{2\pi i \xi \cdot x}\, d\xi =$$

$$\nabla\left(\frac{1}{2\pi i} \int \frac{\hat{X}(\xi) \cdot \xi}{|\xi|^2}\, e^{2\pi i \xi \cdot x}\, d\xi\right)$$

$$\vec{X}_2(x) = \int \frac{\xi \wedge \hat{X}(\xi)}{|\xi|^2} \wedge \xi \; e^{2\pi i x \xi}\, d\xi =$$

$$= \operatorname{rot}\left(\frac{1}{2\pi i} \int \frac{\xi \wedge \hat{X}(\xi)}{|\xi|^2}\, e^{2\pi i x \xi}\, d\xi\right)$$

3d] Vorticity-stream formulation of the N-S equations

In the NS equations the velocity field $v(x,t)$ is divergence-free. If $\omega(x,t) = \mathrm{rot}_x v(x,t)$ is the <u>vorticity</u> we thus have ($v =$

$$v(x,t) = +\int_{\mathbb{R}^3} K(x-y)\, \omega(y,t)\, dy$$

where $K(x)$ is the 3×3 matrix kernel acting

$$K(x)\, h = -\frac{1}{4\pi} \frac{x \times h}{|x|^3}$$

The vorticity $\omega(x,t)$ satisfies the equation

$$\frac{D\omega}{Dt} = \mathcal{D}\omega + \nu \Delta \omega$$

$$\underbrace{\mathcal{D} = \frac{1}{2}(\nabla v + \nabla v^t)}_{\text{deformation matrix}} \quad , \quad \mathcal{D}\omega = \nabla v \cdot \omega$$

We have v in terms of ω. Now we use the results above to compute ∇v

$$[\nabla v]\, h = -\frac{1}{4\pi} \mathrm{p.v.} \int \nabla_x \frac{(x-y) \times \omega(y,t)}{|x-y|^3} h\, dy$$

$$+ \frac{1}{4\pi} \int_{|y|=1} [y \times \omega(x)]\, y \cdot h\, d\sigma = \text{idem} - \frac{1}{3} \omega(x) \times h$$

When computing $\omega \cdot \nabla v$ the second term disappears

$$\mathcal{D}(x) = \text{p.v.} \int P(x-y)\,\omega(y)\,dy$$

> In conclusion both ∇v and $\mathcal{D}$ are obtained from ω through a CZ-operator.

It can be shown that the NS equation are <u>equivalent</u> ~~for~~ to the evolution equation for the vorticity

$$\frac{D\omega}{Dt} = \omega \cdot \nabla v + \nu \Delta \omega$$

in the understanding that

$$v(x,t) = \int_{\mathbb{R}^3} K(x-y)\,\omega(y,t)\,dy$$

$[\nabla v][h] =$ as above.

4th Example Korn's inequality.

$u : \mathbb{R}^d \to \mathbb{R}^d$

$$\nabla u(x) = \left(D_j u_k(x) \right)$$

$$\mathcal{E}(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^t \right) = \text{symmetric part}$$

$$\eta(u) = \nabla u - \mathcal{E}(u) = \frac{1}{2}\left(\nabla u - (\nabla u)^t \right) = \text{antisym}.$$

Korn's inequality establishes that $\eta(u)$ is controled by $\mathcal{E}(u)$.

Again, if $p=2$, we show that for $u \in C_c^\infty(\mathbb{R}^d)^d$

$$\| \eta(u) \|_2 \leq \| \mathcal{E}(u) \|_2$$

by integration by parts. With the notation

$$\nabla \cdot u = \sum_{j=1}^d D_j u_j$$

$$u \cdot \nabla = \sum_{j=1}^d u_i D_i$$

one has pointwise

$$\| \mathcal{E}(u)(x) \|_2^2 - \| \eta(u) \|_2^2 = (\nabla \cdot u)^2 + \nabla \cdot \left((u \cdot \nabla) u - (\nabla \cdot u) u \right)$$

and hence

$$\int |\eta(u)|^2 = \int |\mathcal{E}(u)|^2 - \int (\nabla \cdot u)^2 \leq \int |\mathcal{E}(u)|^2$$

The question arises: which is the operator that maps $\varepsilon(u)$ to $\eta(u)$? Does it satisfy L^p-estimates

$$\|\eta(u)\|_p \leq C_p \|\varepsilon(u)\|_p, \quad 1<p<+\infty.$$

We will see again that a CZ operator arises.

Let us compute in the Fourier transform side, but first we check the following pointwise equality

$$D_\ell D_k u_j = \frac{1}{2}\left(D_k \varepsilon_{\ell j} + D_\ell \varepsilon_{jk} - D_j \varepsilon_{\ell k} \right)$$

We apply again D_ℓ and sum in ℓ

$$\Delta D_k u_j = \frac{1}{2}\left(\sum_\ell D_\ell D_k \varepsilon_{\ell j} + \Delta \varepsilon_{jk} - \sum_\ell D_\ell D_j \varepsilon_{\ell k} \right)$$

hence

$$\Delta\left(\overbrace{D_k u_j - D_j u_k}^{\eta_{kj}} \right) = \sum_\ell \left(D_\ell D_k \varepsilon_{\ell j} - D_\ell D_j \varepsilon_{\ell k} \right)$$

Now we apply FT

$$-4\pi^2 |\xi|^2 \widehat{\eta_{kj}} = \sum_\ell \left(-4\pi^2 \xi_\ell \xi_k \widehat{\varepsilon_{\ell j}} + 4\pi^2 \xi_\ell \xi_j \widehat{\varepsilon_{\ell k}} \right)$$

and so we see

$$\boxed{\eta(u) = CZO(\varepsilon(u))}$$