The Calderon - Zygmund decomposition in $\mathbb{R}^d$, $d \geq 1$, and in homogeneous spaces

<u>Calderon-Zygmund lemma</u>. Let $f \in L^1_{loc}(\mathbb{R}^d)$, $f \geq 0$, $\lambda > 0$

Then there exist disjoint cubes $\{Q_j\}$ such that

(a) $f(x) \leq \lambda$ a.e. $x \notin \bigcup_j Q_j$

(b) $\sum |Q_j| \leq \frac{C}{\lambda} \|f\|_1$

(c) $\frac{1}{|Q_j|} \int_{Q_j} f \leq 2^d \lambda$

(in $d=1$, with the Rising Sun lemma, we had $\fint_{Q_j} f = \lambda$).

We will see two proofs of this, the second one being adaptable to the setting of homogeneous spaces.

<u>The first proof</u> uses a <u>stopping-time argument</u>.

Let $\mathcal{D}_k$ the family of dyadic cubes of size 2^{-k}, and

$$E_k f(x) = \sum_{Q \in \mathcal{D}_k} \left(\fint_Q f \right) 1_Q(x)$$

Note that $\int_{\mathbb{R}^d} E_k f = \int_{\mathbb{R}^d} f$. Let

$$M_d f(x) = \sup_k E_k f(x)$$

the dyadic maximal function. Of course $M_d f(x) \sim Mf(x)$

Let $\Omega = \{x : M_d f(x) > \lambda\} = \bigcup_k \Omega_k$, with

$$\Omega_k = \{x : E_k f(x) > \lambda, \; E_j f(x) \leq \lambda, \; j < k\}.$$

Each Ω_k is a union of certain cubes on $\mathcal{D}_k$. The union of all is our collection of Q_j. For each Q_j

$$\frac{1}{|Q_j|} \int_{Q_j} f > \lambda$$

Hence

$$\sum_k |\Omega_k| = \sum_j |Q_j| \leq \sum_j \frac{1}{\lambda} \int_{Q_j} = \frac{1}{\lambda} \int_{\Omega} f \leq \frac{1}{\lambda} \|f\|_1$$

If $x \notin \bigcup_j Q_j = \bigcup_k \Omega_k = \Omega = \{x : M_d f(x) > \lambda\}$

then $M_d f(x) \leq \lambda$ and by Lebesgue's diff. theorem,

$f(x) \leq \lambda$ a.e. $x \notin \Omega$.

Lastly, if Q_j has been selected means that the mean $\fint_{Q_j} f > \lambda$, but not the mean over "his father" in previous ~~gen~~ ~~dif~~ generation $\tilde{Q}_j$: $\frac{1}{|\tilde{Q}_j|} \int_{\tilde{Q}_j} \leq \lambda$.

Since $|\tilde{Q}_j| = 2^d |Q_j|$

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} f \leq \frac{1}{|Q_j|} \int_{\tilde{Q}_j} f \leq 2^d \lambda . \quad //$$

The second proof uses Whitney's decomposition of arbitrary open sets in $\mathbb{R}^d$. Whitney introduced this in his study of what should be the definition of $C^\infty(F)$, that is

$$C^\infty(F) = C^\infty(\mathbb{R}^d) \Big/ I(F) \qquad F \text{ closed}.$$

$$I(F) = \{ \varphi \in C^\infty(\mathbb{R}^d) : D^\alpha \varphi(x) = 0 \ \forall \alpha \in \mathbb{N}^d, \forall x \in F \}.$$

$C^\infty(F)$ is a space of Whitney jets

Whitney decomposition F closed set in $\mathbb{R}^d \Rightarrow \Omega = F^c =$

$= \bigcup_j Q_j$, with $\mathring{Q}_j$ disjoints, and such that

$$c_1 \delta(Q_j) \le d(Q_j, F) \le c_2 \delta(Q_j)$$

for some constants $c_1(d)$, $c_2(d)$. Here $\delta(Q_j)$ = diameter of Q_j and $d(Q_j, F)$ = distance from Q_j to F

Proof. Let as above $\mathcal{D}_k$ dyadic cubes of size 2^{-k} and

$$\Omega_k = \{ x : c2^{-k} \le d(x,F) \le c\,2^{-k+1} \}$$

Consider $\mathcal{F}_0 = \bigcup_k \{ Q \in \mathcal{D}_k : Q \cap \Omega_k \neq \emptyset \}$

If $x \in Q \cap \Omega_k$, $d(Q,F) \le d(x,F) \le c\,2^{-k+1}$ and

$$d(Q,F) \ge d(x,F) - \delta(Q) > c2^{-k} - \sqrt{d}\,2^{-k} = (c-\sqrt{d})2^{-k}$$

so chosing $c = 2\sqrt{d}$ we get

$$\delta(Q) = \sqrt{d}\,2^{-k} \le d(Q,F) \le 2\sqrt{d}\,2^{-k+1} = 4\,\delta(Q)$$

Obviously, $\Omega = \bigcup_{Q \in \mathcal{F}_0} Q$, the problem is that they are not disjoint.

But if two cubes meet, say Q_1 and Q_2 then one is contained in the other and can be deleted.

So we look for each $Q \in \mathcal{F}_0$ at the maximal cube in $\mathcal{F}_0$ containing it (unique) and keep the maximal ones. //

Second proof of C-Z lemma in $\mathbb{R}^d$: consider the open set $\Omega = \{x : Mf(x) > \lambda\}$, $F = \{x : Mf(x) \leq \lambda\}$

We already know that

$$|\Omega| \leq \frac{c}{\lambda} \|f\|_1$$

Let $\Omega = \bigcup_j Q_j$ the Whitney decomposition.

For each Q_j let $p_j \in F$ / $d(Q_j, F) = d(Q_j, p_j)$

and B_j the smallest ball with center p_j / $B_j \supset \mathring{Q}_j$.

Then $|B_j| \sim |Q_j|$ because $\delta(Q_j) \sim d(Q_j, F)$

$p_j \in F \Rightarrow Mf(x) \leq \lambda \Rightarrow$

$$\lambda \geq \frac{1}{|B_j|} \int_{B_j} f \gtrsim \frac{1}{|Q_j|} \int_{Q_j} f \Rightarrow \frac{1}{|Q_j|} \int_{Q_j} f \leq c\lambda.$$

This proof generalizes to homog. space setting.
(See Harmonic Analysis, E. M. Stein, Princeton UP 1993

Calderon-Zygmund decomposition of L^1 functions

Let $f \in L^1(\mu)$, $\lambda > 0$. Then

$$f = g + b, \quad (g = \text{"good"}, \ b = \text{"bad"}) \text{ with}$$

(a) $|g(x)| \le C\lambda$ a.e. x

(b) $b = \sum_j b_j$, each b_j supported in Q_j, disjoint,

$$\int_{Q_j} b_j \, d\mu = 0, \quad \frac{1}{\mu(Q_j)} \int_{Q_j} |b_j| \, d\mu \le C\lambda$$

(c) $\|g\|_1 \le \|f\|_1$, $\|b\|_1 \le C\|f\|_1$

(d) $g \in L^q(\mu)$ $\forall q > 1$, $\|g\|_q^q \le C\lambda^{q-1}\|f\|_1$

(e) If $f \in L^1(\mu) \cap L^q(\mu)$ then $b \in L^q(\mu)$ and

$b = \sum b_j$ is convergent in $L^q(\mu)$, $\|b\|_q \le \sum_j \|b_j\|_q \le C\|f\|_q$

__Proof__ Apply the above to $|f|$, λ and define

$$g(x) = \begin{cases} f(x), & x \notin \bigcup_j Q_j \\ \frac{1}{|Q_j|}\int_{Q_j} f, & x \in Q_j \end{cases} \qquad b_j(x) = \begin{cases} 0 \\ f(x) - \frac{1}{|Q_j|}\int_{Q_j} f, & x \in Q_j \end{cases}$$

Then (a) follows from

$$\left| \frac{1}{|Q_j|} \int_{Q_j} f \right| \le \frac{1}{|Q_j|} \int_{Q_j} |f| \le C\lambda$$

Also

$$\frac{1}{|Q_j|}\int_{Q_j}|b_j| \le 2\,\frac{1}{|Q_j|}\int_{Q_j}|f| \le 2C\lambda$$

On $\left(\bigcup_j Q_j\right)^c$, $g=f$ and

$$\int_{Q_j}|g| \le \int_{Q_j}|f| \quad \text{, so } \|g\|_1 \le \|f\|_1$$

$$\int_{Q_j}|b_j| \le 2C\lambda\,|Q_j| \Rightarrow \int_{\bigcup Q_j}|b_j| \le 2C\lambda\sum_j|Q_j| \le C\|f\|_1 .$$

For (d), on $\left(\bigcup Q_j\right)^c$, $g=f \to \displaystyle\int_{(\bigcup Q_j)^c}|g|^q \le C\lambda^{q-1}\int_{(\bigcup Q_j)^c}|f|$

on Q_j $\displaystyle\int_{Q_j}|g|^q \le \int_{Q_j}C^q\lambda^q = C^q\lambda^q\,|Q_j| \to$

$$\to \int_{\bigcup Q_j}|g|^q \le C^q\lambda^q\sum|Q_j| \le C\lambda^{q-1}\int|f|$$

For (e), $\displaystyle\int_{Q_j}|b_j(x)|^q \le \int_{Q_j}|f(x)|^q + \int_{Q_j}\left(\frac{1}{|Q_j|}\int_{Q_j}|f|^p d\mu\right) =$

$= 2\displaystyle\int_{Q_j}|f(x)|^q \to$ (also $q=1$).

$\to \|b\|_q \le C\|f\|_q \qquad \displaystyle\Big\|\sum_{j\in I} b_j\Big\|_q^q \le C\int_{\bigcup_{j\in I}Q_j}|f|^q$

Theorem. Let T be a CZO that is,

(a) T is bounded in $L^q(\mu)$, $\|Tf\|_q \leq A \|f\|_q$, $1 < q \leq \infty$

(b) There exists a kernel $K(x,y)$ defined for $x \neq y$ such that if $f \in L^q(\mu)$ has compact support and $x \notin \operatorname{spt} f$

$$Tf(x) = \int f(y) K(x,y) \, d\mu(y)$$

(c) The kernel K satisfies: there exists $c > 1$ and A such that

$$\int_{B(\bar{y}, c\delta)^c} |K(x,y) - K(x,\bar{y})| \, d\mu(x) \leq A, \quad \forall y \in B(\bar{y}, \delta)$$

Then:

(1) T is of weak type $(1,1)$ that is, for $f \in L^1(\mu) \cap L^q(\mu)$

$$\mu\{x : |Tf(x)| > \lambda\} \leq \frac{C}{\lambda} \|f\|_1$$

(2) T is of strong type (p,p) for $1 < p \leq q$

$$\|Tf\|_p \leq C_p \|f\|_p, \quad f \in L^q(\mu) \cap L^p(\mu)$$

If additionally (in case $q < +\infty$)

$$\int_{B(\bar{x}, c\delta)^c} |K(x,y) - K(\bar{x},y)| \, d\mu(y) \leq A \quad \rho(x,\bar{x}) \leq \delta$$

then T is of strong type (p,p), $1 < p < +\infty$.

Proof. We prove (a) exactly as we did with the Hilbert transform. Fix $f \in L^1(\mu) \cap L^q(\mu)$, we will prove

$$\mu\{|Tf| > c\lambda\} \leq \frac{C}{\lambda}\|f\|_1$$

for some suitably chosen c. Let $f = g + b$ the decomposition of f at level λ.

$$g, b \in L^q \qquad Tf = Tg + Tb, \quad Tb = \sum_j Tb_j$$

$$\mu\{|Tf| > c\lambda\} \leq \mu\{|Tg| \geq \tfrac{c\lambda}{2}\} + \mu\{|Tb| \geq \tfrac{c\lambda}{2}\} \stackrel{def}{=} I + II$$

Estimate of I:

In case $q = +\infty$, $\|Tg\|_\infty \leq A\|g\|_\infty \leq A'\lambda$ so choosing $c > 2A'$ this term does not appear.

In case $q < +\infty$, $\|Tg\|_q \leq A\|g\|_q \leq (C\lambda^{q-1}\|f\|_1)^{1/q}$

$$\mu\{|Tg| \geq \tfrac{c}{2}\lambda\} \leq C\lambda^{-q}\|Tg\|_q^q \leq C\lambda^{-q}\lambda^{q-1}\|f\|_1 = \frac{C}{\lambda}\|f\|_1$$

Estimate of II: Let $Q_j = B_j(\bar{y}_j, \delta_j)$ and put $Q_j^* = B_j(\bar{y}_j, c\delta_j)$ where c is in hypothesis (c). We have

$$\sum \mu(Q_j^*) \leq C \sum \mu(B_j) \leq \frac{C}{\lambda}\|f\|_1$$

so it is enough to prove

$$\mu\{x : x \notin \bigcup Q_j^* : |Tb| \geq \tfrac{c\lambda}{2}\} \leq \frac{C}{\lambda}\|f\|_1$$

which in turn follows from

$$\int_{(\bigcup Q_j^*)^c} |Tb(x)|\, d\mu(x) \leq C\|f\|_1$$

Using $|Tb| \le \sum_j |Tb_j|$ the RHS is $\le$

$$\sum_j \int_{(Q_j^*)^c} |T b_j(x)| \, d\mu(x)$$

Now for $x \notin Q_j^*$

$$Tb_j(x) = \int_{Q_j} K(x,y) b_j(y) \, d\mu(y) =$$

$$= \int_{Q_j} (K(x,y) - K(x,\bar{y}_j)) b_j(y) \, d\mu(y) \quad \left(\text{because } \int_{Q_j} b_j \, d\mu = 0\right)$$

$$|Tb_j(x)| \le \int_{Q_j} |K(x,y) - K(x,\bar{y}_j)| \, |b_j(y)| \, d\mu(y)$$

$$\int_{(Q_j^*)^c} |Tb_j(x)| \, d\mu(x) \overset{\text{Fubini}}{\le} \int_{Q_j} |b_j(y)| \left\{ \int_{(Q_j^*)^c} |K(x,y) - K(x,\bar{y}_j)| \, d\mu(x) \right\} d\mu(y)$$

$$\underset{\text{hyps (c)}}{\le} A \int_{Q_j} |b_j(y)| \, d\mu(y)$$

Summing in j we are done, using $\sum \|b_j\|_1 \le c \|f\|_1$ //

Remark 1. In the theorem, nothing is assumed on how T is defined in $L^q(\mu)$. In the examples we saw, $q=2$ and T is bounded in $L^2(\mathbb{R}^d)$ because it was of the form $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$ with a bounded multiplier m, or was given by integration by parts.

The question we face in next lectures is precisely to give criteria so that T is bounded on $L^q(\mu)$ (usually $q=2$).

Remark 2. The kernel $K(x,y)$ does not determine T, that is, $K\equiv 0$ does not imply $T=0$. For instance, a multiplication operator $Tf(x) = a(x) f(x)$ has $K=0$. In fact, if $K\equiv 0$ then Tf is a multiplication operator, $Tf(x) = a(x) f(x)$ for some $a \in L^\infty$ (exercise)

Definition. A standard CZO operator is one for which K satisfies (using the quasidistance $\rho(x,y)$)

(a) $$|K(x,y)| \le \frac{C}{V(x,y)}, \quad V(x,y) = \inf\{\mu(B(y,\delta)),\ x\in B(y,\delta)\}$$

(b) $$|K(x,y) - K(x,\bar{y})| \le \eta\left(\frac{\rho(y,\bar{y})}{\rho(x,\bar{y})}\right)\frac{1}{V(x,y)}, \quad \rho(x,\bar{y}) \ge c\,\rho(y,\bar{y})$$

(c) $$|K(x,y) - K(\bar{x},y)| \le \eta\left(\frac{\rho(x,\bar{x})}{\rho(\bar{x},y)}\right)\frac{1}{V(x,y)}, \quad \rho(y,\bar{x}) \ge c\,\rho(x,\bar{x})$$

$$\int_0 \frac{\eta(s)}{s}\,ds < +\infty.$$

Example. The Cauchy transform on a Lipschitz curve

$\Gamma = (t, A(t))$, $t \in \mathbb{R}$, curve in $\mathbb{R}^2$ with

A lipschitz : $|A(t) - A(s)| = O(|t-s|)$, $a = A'$

$$C_\Gamma f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, dz \quad , z \notin \Gamma \text{ (Cauchy transform)}$$

Recall the Plemej formula:

$$\lim_{\varepsilon \to 0} C_\Gamma f(x + iA(x) + i\varepsilon) = \frac{1}{2} f(x) + \frac{1}{2} Tf(x)\, i$$

$$+ \quad Tf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|x-t|>\varepsilon} \frac{f(t + \cancel{iA(t)})(1 + i a(t))}{x - t + i(A(x) - A(t))}\, dt$$

$$Cf(x) = \lim_{\varepsilon \to 0} \int_{|x-t|>\varepsilon} \frac{f(t)}{(x-t) + i(A(x) - A(t))}\, dt$$

is a CZO with kernel

$$K(x,y) = \frac{1}{(x-y) + i(A(x) - A(y))}$$

that satisfies standard assumptions with $\gamma(s) = s$

To prove that C is bounded in $L^2(\mathbb{R})$ is <u>hard</u>.

Example. Closely related to the above are the Calderon commutators

$$\mathcal{C}_k f(x) = \lim_{\varepsilon} \int_{|x-y|>\varepsilon} \left(\frac{A(x)-A(y)}{x-y} \right)^k \frac{f(y)}{x-y}\, dy$$

also standard CZO.

The truncated operators for a standard CZO

If T is standard, $|K(x,y)| \lesssim \dfrac{1}{V(x,y)}$ and

for fixed x

$$\mu\{ y : |K(x,y)| > \lambda \} \le \mu\{ y : V(x,y) < \tfrac{c}{\lambda} \}$$

But $V(x,y) < \frac{c}{\lambda}$ means that there exists a ball $B(x,\delta)$ with $y \in B(x,\delta)$ and $\mu(B(x,\delta)) < \frac{c}{\lambda}$ $\Rightarrow$

$$\mu\{ y : |K(x,y)| > \lambda \} \le \frac{c}{\lambda}$$

Then we can define

$$T_\varepsilon f(x) = \int_{\rho(x,y) \ge \varepsilon} K(x,y) f(y)\, dy \quad , \ f \in L^q(\mu) \ , q>1$$

$$= \int K_\varepsilon(x,y) f(y)\, dy$$

One can prove that the $T_\varepsilon, K_\varepsilon$ satisfy the assumptions of the main theorem with constants independent of ε, and hence

$$\|T_\varepsilon f\|_p \le C_p \|f\|_p \quad , f \in L^q(\mu) \cap L^p(\mu)$$

$$\mu\{|T_\varepsilon f| > \lambda\} \le \frac{C}{\lambda} \|f\|_1 \quad , f \in L^q(\mu) \cap L^1(\mu)$$

The $\{T_\varepsilon\}_\varepsilon$ being uniformly bounded operators in $L^p(\mu)$ there exists a sequence ε_j such that

$$T_0 f = \lim T_{\varepsilon_j} f$$

exists $\forall f \in L^p(\mu)$. Not necessarily $\lim_\varepsilon T_\varepsilon f$ exists though (see example later). Then T and T_0 have the same K, so

$$Tf = T_0 f + a f$$

To study the existence of $\lim_\varepsilon T_\varepsilon f(x)$ one introduces

$$T_* f(x) = \sup_\varepsilon |T_\varepsilon f(x)|.$$

One can then prove (details in Stein's book pg 34)

<u>Thm</u>. For all q, $0 < q \le 1$

$$T^* f(x) \le C(q) \left[\left(M(|Tf|^q(x))\right)^{1/q} + M f(x) \right]$$

This implies

<u>Theorem</u>. The maximal operator $T_* f$ of a standard CZO operator satisfies

$$\|T_* f\|_p \leq C_p \|f\|_p \qquad 1 < p \leq q$$

$$\mu\{|T_* f| > \lambda\} \leq \frac{C}{\lambda} \|f\|_1$$

<u>Proof</u>. The first assertion follows from ($\eta = 1$), and the boundedness of T, M in $L^p(\mu)$.

For the second one,

$$\mu\{|T_* f| > \lambda\} \leq \mu\{|M(Tf)^\eta| > \lambda^\eta\} + \mu\{|Mf| > \lambda\}$$

The second term above is OK because M is weak $(1,1)$. For the first term we argue as follows: we know that Tf is in weak L^1. This means that $|Tf|^\eta$ is in the Lorentz space $L^{1/\eta,\infty}$. Then so is $M(Tf)^\eta$ (M is bounded in all Lorentz spaces) and we are done. //

The existence of

$$\lim_{\varepsilon} T_\varepsilon f$$

both in L^p-norm, $1 < p < +\infty$ or pointwise a.e. is then reduced to the existence in a <u>dense</u> subspace.

The Calderon - Zygmund convolution operators in $\mathbb{R}^d$

A bounded operator $T: \underbrace{L^q(\mathbb{R}^d) \to L^q(\mathbb{R}^d)}_{\text{invariant by translations}}$ is one of the form $T\varphi = \varphi * \mu$, for some tempered distribution μ. Application of the CZ theorem gives that if

(a) $\hat{\mu} = m \in L^\infty(\mathbb{R}^d)$ (so that is bounded in $L^2(\mathbb{R}^d)$)

(b) μ coincides on $\mathbb{R}^d \setminus \{0\}$ with a locally integrable function k such that

$$\int_{|x| \geq 2|y|} |k(x-y) - k(x)| \, dx \leq C \quad \text{(Hörmander condition)}$$

then $T\varphi = \varphi * \mu$ satisfies

$$\|Tf\|_p \leq C_p \|f\|_p \qquad f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$$

$$|\{|Tf| > \lambda\}| \leq \frac{C}{\lambda} \|f\|_1 \quad , \quad f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$$

We noticed before that if $k \in C^1(\mathbb{R}^d \setminus \{0\})$ and

$$|\nabla k(x)| \leq \frac{C}{|x|^{d+1}}$$

then the Hörmander condition holds.

In this situation, $\mu \in S'$, $\mu = k \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ on $\mathbb{R}^d \setminus \{0\}$, μ is not determined by k (for instance we can add to μ a tempered distribution supported on $\{0\}$). But it is natural to look at those given only by k

Proposition Let $k \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ such that

(a) $$\left| \int_{a<|x|<b} k(x)\,dx \right| \le A \qquad 0<a<b<+\infty.$$

(b) $$\int_{a<|x|<2a} |k(x)|\,dx \le A$$

(c) $$\int_{|x|>2|y|} |k(x-y)-k(x)|\,dx \le A \qquad \forall y$$

Let $k_{\varepsilon,R}(x) = k(x)\,\mathbb{1}_{\{\varepsilon<|x|<R\}}(x)$. Then

$$\left|\widehat{k_{\varepsilon,R}}(\xi)\right| \le C$$

with C independent of ε, R.

Notice that (b) is equivalent to

$$\int_{|x|<a} |x|\,|k(x)|\,dx \le B a$$

Indeed: $$\int_{|x|<a} |x|\,|k(x)|\,dx = \sum_{j=0}^{\infty} \int_{2^{-j-1}a<|x|<2^{-j}a} |x|\,|k(x)|\,dx \le$$

$$\le \sum_{j=0}^{\infty} 2^{-j}a \int_{2^{-j-1}a<|x|<2^{-j}a} |k(x)|\,dx \le \sum_{j=0}^{\infty} 2^{-j}a\,A = \underbrace{2A}_{B}\,a$$

and conversely

$$\int_{a<|x|<2a} |k(x)|\,dx \leq \int_{|x|\leq 2a} \frac{|x|}{a} |k(x)|\,dx \leq 2C$$

Proof of proposition:

$$\widehat{k}_{\varepsilon,R}(\xi) = \int_{\varepsilon<|x|<R} k(x)\, e^{-2\pi i x\cdot\xi}\,dx = \int_{\varepsilon<|x|<|\xi|^{-1}} + \int_{|\xi|^{-1}<|x|<R}$$

$$\overset{\text{def}}{=} I_1 + I_2 \qquad (\text{if } \varepsilon < |\xi|^{-1} < R \text{ otherwise only one int})$$

Then

$$I_1 = \int_{\varepsilon<|x|<|\xi|^{-1}} k(x)\,dx + \int_{\varepsilon<|x|<|\xi|^{-1}} k(x)\left(e^{-2\pi i x\cdot\xi} - 1\right)dx$$

$$|I_1| \leq \left|\int_{\varepsilon<|x|<|\xi|^{-1}} k(x)\,dx\right| + 2\pi|\xi| \int_{|x|<|\xi|^{-1}} |x|\,|k(x)|\,dx \lesssim A+B$$

For I_2, let $z = \frac{1}{2}\frac{\xi}{|\xi|^2}$, so that $e^{-2\pi i z\cdot\xi} = -1$; then

$$I_2 = -\int_{|\xi|^{-1}<|x-z|<R} k(x-z)\, e^{-2\pi i x\cdot\xi}\,dx$$

$$2I_2 = \int_{|\xi|^{-1}<|x|<R} k(x)\, e^{-2\pi i x\cdot\xi}\,dx - \int_{|\xi|^{-1}<|x-z|<R} k(x-z)\, e^{-2\pi i x\cdot\xi}\,dx$$

$$2|J_2| \le \int_{|\xi|^{-1}<|x|} |k(x)-k(x-z)|\,dx + \int_{\frac{1}{2}|\xi|^{-1}<|x|<\frac{3}{2}|\xi|^{-1}} |k(x)|\,dx +$$

$$+ \int_{R-\frac{1}{2}|\xi|^{-1}<|x|<R+\frac{1}{2}|\xi|^{-1}} |k(x)|\,dx \le 5A. \quad //$$

Notice also that condition (b) follows from

$$|k(x)| \le \frac{A}{|x|^d}$$

Suppose now that the limit

$$\lim_{\varepsilon} \int_{\varepsilon<|x|<1} k(x)\,dx = b$$

exists. Then it exists the tempered distribution

$$\langle \text{v.p. } k, \varphi\rangle = \lim_{\substack{\varepsilon\to 0\\ R\to+\infty}} \int_{\varepsilon<|x|<R} k(x)\varphi(x) = \lim_{\varepsilon\to 0} \int_{\varepsilon<|x|} k(x)\varphi(x)$$

$$= \int_{1<|x|} k(x)\varphi(x) + \int_{|x|<1} k(x)(\varphi(x)-\varphi(0)) + b\,\varphi(0))$$

and the operator $T\varphi = \text{v.p. } k * \varphi$ satisfies all conditions. ~~[illegible]~~ because

$$\widehat{\text{vp}\,k} = \lim_{\varepsilon,R} \widehat{k_{\varepsilon,R}} \in L^\infty$$

Theorem. If $k \in L^1_{loc}(\mathbb{R}^d \setminus 0)$ satisfies conditions (a), (b), (c) of proposition above and moreover the limit $\lim_\varepsilon \int_{\varepsilon<|x|<1} k(x)\,dx$ exists, then

$$T\varphi(x) = \lim_\varepsilon \int_{|x-y|>\varepsilon} \varphi(y)\, k(x-y)\, dy, \quad \varphi \in S(\mathbb{R}^d)$$

can be extended to a bounded operator in $L^p(\mathbb{R}^d)$, $1<p<\infty$, and weak (1,1). ~~The maximal operator~~ If

$$|k(x)| \leq \frac{C}{|x|^d}$$

the maximal operator

$$T_* f(x) = \sup_\varepsilon |T_\varepsilon f(x)|$$

satisfies too strong (p,p) estimates and a weak (1,1) estimate, so that

$$Tf(x) = \lim_\varepsilon T_\varepsilon f(x)$$

both in $L^p(\mathbb{R}^d)$ (or in measure) and pointwise.

These operators are called <u>singular convolution operators</u>.

Singular convolution operators which commute with dilations

We consider again the dilation operators

$$(\rho_\lambda f)(x) = f(\lambda x), \quad \lambda > 0$$

If an operator T which commutes with translations (that is, a convolution operator with kernel $k(x-y)$) is to commute too with dilations, then k must be homogeneous of degree $-d$, $k(\lambda x) = \lambda^{-d} k(x)$

This is the same as saying that

$$k(x) = \frac{\Omega(x')}{|x|^d}, \quad x' = \frac{x}{|x|}$$

The condition (2) of the previous theorem

$$\left| \int_{a<|x|<b} k(x)\, dx \right| \leq A \iff \int_{S^{d-1}} \Omega(x')\, d\sigma(x') = 0.$$

while condition $|k(x)| = O\left(\frac{1}{|x|^d}\right)$ says Ω bounded.

Let us state a condition on Ω so that k satisfies the Hormander condition

Prop. If $\Omega(u)$, $u \in S^{d-1}$ is continuous and the modulus of continuity

$$\omega(\delta) = \sup\{ |\Omega(u_1) - \Omega(u_2)| : |u_1 - u_2| \leq \delta \}$$

satisfies a Dini-condition

$$\int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta < \infty$$

then $k(x) = \Omega(x')/|x|^d$ satisfies the Hormander condition.

Proof.

$$|k(x-y)-k(x)| = \left| \frac{\Omega((x-y)')}{|x-y|^d} - \frac{\Omega(x')}{|x|^d} \right| \leq$$

$$\leq \frac{|\Omega((x-y)') - \Omega(x')|}{|x-y|^d} + |\Omega(x')| \left| \frac{1}{|x-y|^d} - \frac{1}{|x|^d} \right|$$

Ω is bounded, hence the integral of the second term over $|x| > 2|y|$ is finite, bounded uniformly in y.

$$|(x-y)' - x'| = \left| \frac{x-y}{|x-y|} - \frac{x}{|x|} \right| = \left| \frac{|x|(x-y) - x|x-y|}{|x||x-y|} \right|$$

$$= \left| \frac{(|x| - |x-y|)(x-y) + |x-y|\, y}{|x||x-y|} \right| \leq 2\frac{|y|}{|x|}.$$

$$\int_{|x|\geq 2|y|} \frac{\omega\left(2\frac{|y|}{|x|}\right)}{|x-y|^d} dx \leq C \int_{|x|\geq 2|y|} \frac{\omega\left(2\frac{|y|}{|x|}\right)}{|x|^d} dx \leq C \int \frac{\omega(2\delta)}{\delta} d\delta$$

In this case

$$b = \lim_{\varepsilon} \int_{\varepsilon < |x| < 1} k(x)\,dx = 0.$$

Hence:

<u>Theorem</u>. If $\Omega \in C(S^{d-1})$ has a Dini modulus of continuity, and ~~[illegible]~~ has integral zero, the operator

$$T\varphi(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \varphi(x-y) \frac{\Omega(y')}{|y|^d}\, dy, \quad \varphi \in S$$

~~is~~

$$= \text{p.v.} \int f(x-y) \frac{\Omega(y')}{|y|^d}\, dy$$

exists a.e. for $f \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$ and satisfies a weak $(1,1)$ estimate and strong (p,p), $1<p<\infty$.

In this case we can compute explicitly the multiplier

$m = \widehat{\text{vp}\cdot K}$:

<u>Prop</u>. ~~[illegible]~~ The Fourier transform of $\text{vp}\, \frac{\Omega(x')}{|x|^d}$ is

$$m(\xi) = \int_{S^{d-1}} \Omega(u) \left[\log \frac{1}{|u \cdot \xi'|} - i\frac{\pi}{2} \operatorname{sgn}(u \cdot \xi') \right] d\sigma(u)$$

<u>Proof</u> By homogeneity of the FT, ~~m~~ is homogeneous of degree zero, that is, radial. We may assume $|\xi| = 1$.

$$m(\xi) = \lim_{\varepsilon \to 0} \int_{\varepsilon < |y| < 1/\varepsilon} \frac{\Omega(y')}{|y|^d} e^{-2\pi i y\cdot\xi}\, dy =$$

$$= \lim_{\varepsilon} \int_{S^{d-1}} \Omega(u) \left(\int_{\varepsilon}^{1} (e^{-2\pi i r u\cdot\xi} - 1) \frac{dr}{r} + \int_{1}^{1/\varepsilon} e^{-2\pi i r u\cdot\xi} \frac{dr}{r} \right) d\sigma(u)$$

$$\longrightarrow \quad m(\xi) = I_1 - i\, I_2$$

$$I_1 = \lim_{\varepsilon} \int_{S^{d-1}} \Omega(u) \left[\int_{\varepsilon}^{1} \left(\cos 2\pi r (u\cdot\xi) - 1\right) \frac{dr}{r} + \int_{1}^{1/\varepsilon} \cos 2\pi r (u\cdot\xi) \frac{dr}{r} \right] d\sigma$$

$$I_2 = \lim_{\varepsilon} \int_{S^{d-1}} \Omega(u) \left[\int_{\varepsilon}^{1/\varepsilon} \sin 2\pi r (u\cdot\xi) \cdot \frac{dr}{r} \right] d\sigma(u)$$

In the dr integral, $s = 2\pi r (u\cdot\xi) \rightarrow$ (in I_2)

$$\int_{2\pi|u\cdot\xi|\varepsilon}^{2\pi|u\cdot\xi|\frac{1}{\varepsilon}} \sin s \;\; \mathrm{sgn}\,(u\cdot\xi)\, \frac{ds}{s} =$$

$$= \mathrm{sgn}(u\cdot\xi) \int_{2\pi|u\cdot\xi|\,\varepsilon}^{2\pi|u\cdot\xi|\frac{1}{\varepsilon}} \frac{\sin s}{s}\, ds \rightarrow \mathrm{sgn}\,(u\cdot\xi) \int_0^{\infty} \frac{\sin s}{s}\, ds =$$

$$= \frac{\pi}{2} \;\; \mathrm{sgn}(u\cdot\xi)$$

In I_1 the inner integral is

$$\int_{2\pi|u\cdot\xi|\,\varepsilon}^{1} (\cos s - 1)\, \frac{ds}{s} + \int_{1}^{2\pi|u\cdot\xi|\frac{1}{\varepsilon}} \cos s \;\frac{ds}{s} - \int_{1}^{2\pi|u\cdot\xi|} \frac{ds}{s}$$

$$\longrightarrow \underbrace{\int_0^1 (\cos s - 1)\, \frac{ds}{s} + \int_1^{\infty} \frac{\cos s}{s}\, ds - \log 2\pi}_{} - \log|u\cdot\xi|$$

integrated against $\Omega(u)$ gives zero. //

The Riesz transforms

The Riesz transforms R_j correspond to the choice $\Omega_j(x) = c_d \frac{x_j}{|x|}$, that is

$$R_j f(x) = c_d \text{ p.v.} \int \frac{y_j}{|y|^{d+1}} f(x-y)\,dy \quad , j = 1, \dots, d.$$

The constant c_d is chosen so that

$$\widehat{R_j f}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi)$$

(so that $R_1 = H$ if $d=1$). Here is the computation. We saw that as distributions

$$D_j |x|^{-d+1} = (1-d) \text{ p.v.} \frac{x_j}{|x|^{d+1}}$$

and so the FT of p.v. $\frac{x_j}{|x|^{d+1}}$ is

$$\frac{1}{1-d} \left(D_j |x|^{-d+1} \right)^\wedge (\xi) = \frac{1}{1-d} 2\pi i \xi_j \left(|x|^{-d+1} \right)^\wedge (\xi)$$

$$= -i \frac{\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \frac{\xi_j}{|\xi|}$$

so $c_d = \dfrac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}}$

Note that $\sum_j R_j^2 = -Id$.