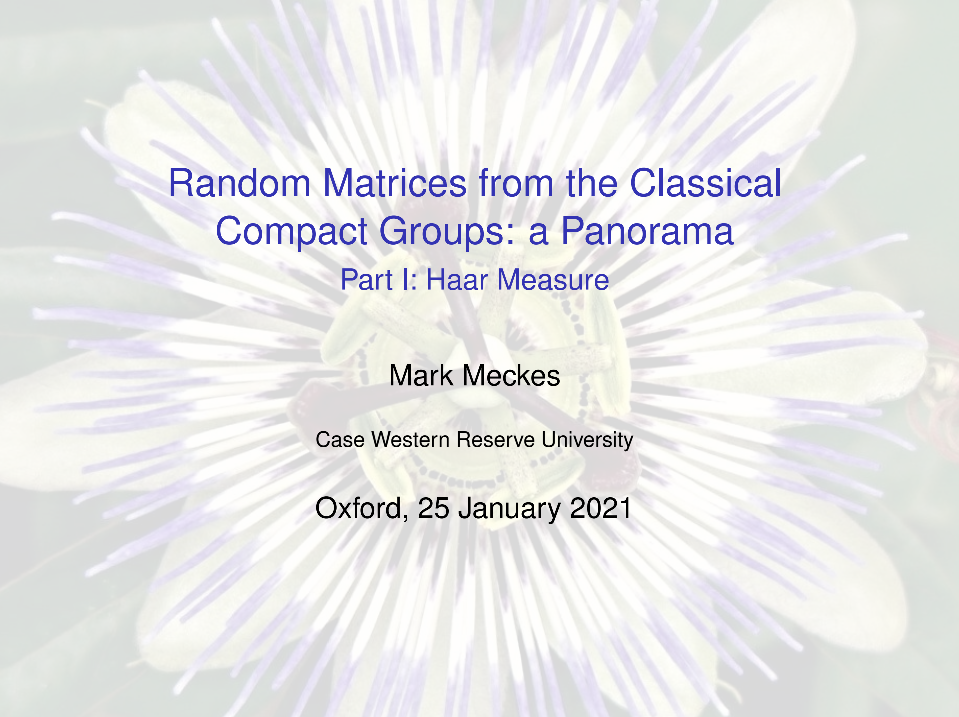

Random Matrices from the Classical Compact Groups: a Panorama

Part I: Haar Measure

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(Tentative) outline of this series

1. (Today) Haar measure on the classical compact groups
2. Orthogonal v. Gaussian random matrices
3. Concentration of measure
4. Applications of measure concentration
5. Eigenvalue distributions: exact formulas
6. Eigenvalue distributions: asymptotics
7. Eigenvalue distributions: inequalities
8. More (or more detail)?

Primary reference:

Elizabeth Meckes, *The Random Matrix Theory of the Classical Compact Groups*, Cambridge, 2019.

www.case.edu/artsci/math/esmeckes/Haar_book.pdf

What colored text means:

Red means stop.

Green means keep going.

Violet and blue are to make things more colorful.

Why study random matrices?

Why study random *anything*?

- Knowing what's "typical"
- Modeling uncertainty
- Randomized algorithms
- Probabilistic existence proofs
- Beautiful mathematics and unexpected connections

The classical compact groups

Our main characters are two cousins:

- The orthogonal group $\mathbb{O}(n) = \left\{ U \in M_n(\mathbb{R}) \middle| UU^T = I_n \right\}$
- The unitary group $\mathbb{U}(n) = \{ U \in M_n(\mathbb{C}) | UU^* = I_n \}$

They look a lot alike, but there are subtle and important differences.

Rule of thumb:

- $\mathbb{O}(n)$ is mostly easier to work with geometrically.
- $\mathbb{U}(n)$ is mostly easier to work with algebraically.

The classical compact groups

We'll also meet their kid sisters:

- $\mathbb{SO}(n) = \{U \in \mathbb{O}(n) | \det U = 1\}$.
- $\mathbb{SO}^-(n) = \{U \in \mathbb{O}(n) | \det U = -1\}$.
- $\mathbb{SU}(n) = \{U \in \mathbb{U}(n) | \det U = 1\}$.

$\mathbb{O}(n) = \mathbb{SO}(n) \sqcup \mathbb{SO}^-(n) = \mathbb{SO}(n) \rtimes \{\pm 1\}$
$\mathbb{U}(n) = \mathbb{SU}(n) \rtimes \mathbb{U}(1)$ is connected.

The classical compact groups

The weird uncle no one talks about:

The compact symplectic group

$$\mathbb{S}\mathrm{p}(n) = \{U \in \mathbb{U}(2n) | UJ_nU^* = J_n\},$$

where $J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

This is really the unitary group of $n \times n$ quaternionic matrices.

Structures on the classical compact groups

Each of these sets is:

- a compact Lie group, and hence:
  - a Riemannian manifold and
  - a topological group;
- a subset of a Euclidean space:
  - $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$, with the Frobenius / Hilbert–Schmidt inner product $\langle A, B \rangle = \mathrm{Tr}\, AB^*$.

Structures on the classical compact groups

Each has two different metric space structures:

- the geodesic metric $d_g(U, V)$ from the manifold structure,
- the Euclidean metric $d_{HS}(U, V) = \sqrt{\langle U - V, U - V \rangle}$.

Conveniently,

$$d_{HS}(U, V) \leq d_g(U, V) \leq \frac{\pi}{2} d_{HS}(U, V)$$

for $U, V \in \mathbb{U}(n)$.

Haar measure

Theorem

If $\mathbb{G}$ is a compact topological group, there is a unique probability measure μ which is invariant: $\mu(V\mathcal{A}) = \mu(\mathcal{A})$ for every $\mathcal{A} \subseteq \mathbb{G}$ and $V \in \mathbb{G}$.
This measure also satisfies $\mu(\mathcal{A}V) = \mu(\mathcal{A}) = \mu(\mathcal{A}^{-1})$.

μ is called the Haar measure on $\mathbb{G}$.

That is: there is a unique notion of a random $U \in \mathbb{G}$ with the property that for each fixed $V \in \mathbb{G}$, $VU \sim U$.

It is then also true that $UV \sim U^{-1} \sim U$.

Haar measure

Typical interpretation:

There is one unique “reasonable” notion of a random $U \in \mathbb{G}$.

Specialized to the classical compact groups $\mathbb{G}$, that’s what these lectures are about.

But: don’t necessarily take that interpretation too seriously!

A bit of terminology

For these lectures:

Random $U \in \mathbb{G}$ always means chosen according to Haar measure.

A random $U \in \mathbb{U}(n)$ is sometimes called the Circular Unitary Ensemble (CUE).

Warning:

The Circular Orthogonal Ensemble and Circular Symplectic Ensemble are not the same as random matrices in $\mathbb{O}(n)$ or $\mathbb{S}\mathrm{p}(n)$.

What is Haar measure, really?

The uniqueness part of the theorem means:

If you come up with some way to pick a random $U \in \mathbb{G}$,
and it turns out to be invariant,
then it's our way to pick a random $U \in \mathbb{G}$.

There are a bunch of different ways to do that!

Riemannian construction

If $\mathbb{G}$ is a compact Lie group, then the normalized Riemannian volume form

$$\mu(\mathcal{A}) = \frac{\int_{\mathcal{A}} d\mathrm{vol}_g}{\int_{\mathbb{G}} d\mathrm{vol}_g}$$

is invariant.

So it's Haar measure!

Euclidean construction

Pick $U_1 \in S^{n-1}$ uniformly,
then pick $U_2 \in S^{n-1} \cap U_1^{\perp}$ uniformly,
then pick $U_3 \in S^{n-1} \cap \{U_1, U_2\}^{\perp}$ uniformly,
⋮
then pick $U_n \in S^{n-1} \cap \{U_1, \ldots, U_{n-1}\}^{\perp}$ uniformly.

Then $U = \begin{bmatrix} | & \cdots & | \\ U_1 & \cdots & U_n \\ | & \cdots & | \end{bmatrix} \in \mathbb{O}(n)$ is Haar-distributed.

Inductive construction

Say we already know how to generate a random $U_{n-1} \in \mathbb{O}(n-1)$.

Can we leverage that?

Pick $X \in S^{n-1}$ uniformly and independently of U_{n-1}.
Pick $M \in \mathbb{O}(n)$ independently of U with first column X.

Then $U = M \begin{bmatrix} 1 & 0 \\ 0 & U_{n-1} \end{bmatrix} \in \mathbb{O}(M)$ is Haar-distributed.

Gauss–Gram–Schmidt construction

If $M \in M_n(\mathbb{C})$ is nonsingular and we perform the Gram–Schmidt process on the columns of M, we get a $U \in \mathbb{U}(n)$.

If M is random and invariant w.r.t. unitary multiplication, then so is U, so U is Haar-distributed.

Easiest way to do that:

Let M be a Gaussian random matrix: independent standard Gaussian entries.

Gauss–Gram–Schmidt construction

We can summarize that construction via a matrix factorization (handy for computer simulation):

Let M be a Gaussian random matrix and let

$$M = QR$$

be the QR decomposition of M, then take $U = Q$.

BUT:

QR decompositions are not unique!

If your QR decomposition is not based on Gram–Schmidt then Q won't necessarily by Haar-distributed.

Gaussian polar decomposition construction

Every nonsingular $M \in M_n(\mathbb{C})$ has a unique polar decomposition

$$M = UP$$

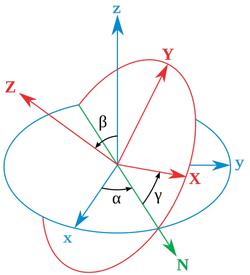
where $U \in \mathbb{U}(n)$ and $P \in M_n(\mathbb{C})$ is positive definite.

$P = (M^* M)^{1/2}$ and $U = MP^{-1}$.

If M is a Gaussian random matrix then U is Haar-distributed.

Euler angles

A rotation $U \in \mathbb{SO}(n)$ can be specified as a product of rotations in 2-dimensional planes through Euler angles θ_j^k, $1 \leq k \leq n-1$, $1 \leq j \leq k$.



A Haar-distributed $U \in \mathbb{SO}(n)$ corresponds to picking the θ_j^k independently with densities $\propto \sin^{j-1}(\theta_j^k)$.

Permutation invariance

Invariance is enough to prove basic properties without a concrete construction.

If P, Q are permutation matrices then

$$PUQ \sim U.$$

$\Longrightarrow$

- All entries of U are identically distributed.
- All $k \times \ell$ submatrices of U are identically distributed.

Moments of entries

We can compute a lot about joint distributions of entries just from symmetry and knowing the distribution of a single column:

If $U \in \mathbb{O}(n)$ is Haar-distributed:

- $$\mathbb{E}u_{11}^2 = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}u_{j1}^2 = \frac{1}{n}$$
- $$\mathbb{E}u_{11}^2 u_{21}^2 = \frac{1}{n(n+2)}$$
- $$\mathbb{E}u_{11}^2 u_{22}^2 = \mathbb{E}u_{11}^2\left(\frac{1}{n-1}\sum_{j=2}^{n}u_{j2}^2\right) = \frac{1}{n-1}\mathbb{E}u_{11}^2(1-u_{12}^2)$$
  $$= \frac{n+1}{(n-1)n(n+2)}$$

Observation: $\left|\mathrm{Cov}(u_{ij}^2 u_{kl}^2)\right| \leq \frac{C}{n^3}$ if $(i,j) \neq (k,\ell)$.

Moments of entries

It's possible to be totally systematic:

Theorem (Weingarten calculus, Collins)

If $U \in \mathbb{U}(n)$ is Haar-distributed, then

$$\mathbb{E} u_{i_1,j_1} \cdots u_{i_k,j_k} \overline{u_{i'_1,j'_1} \cdots u_{i'_k,j'_k}} = \sum_{\sigma,\tau \in S_n} \mathbb{1}_{i_\sigma = i', j_\tau = j'} \mathrm{Wg}^U(\sigma^{-1}\tau, n),$$

where the Weingarten function Wg^U is a sum of terms related to the irreducible representations of S_n.

More complicated versions exist for $\mathbb{O}(n)$ and $\mathbb{Sp}(n)$.

Joint distribution of entries

Theorem (Eaton)

Let $1 \leq p \leq q$ and $p + q \leq n$. Then the upper-left $p \times q$ submatrix M of a random $U \in \mathbb{O}(n)$ has a density proportional to

$$\det(I_q - M^T M)^{\frac{n-p-q-1}{2}}$$

on $\left\{ M \in M_{p \times q}(\mathbb{R}) \middle| \lambda_{\max}(M^T M) < 1 \right\}$.

Rough outline of Eaton's proof:

- Both descriptions of M give $M^T M$ a matrix-variate beta distribution.
- The distribution of M is determined by the distribution of $M^T M$ (invariant theory).

Other proofs, and versions for $\mathbb{U}(n)$ and $\mathbb{Sp}(n)$, are also known.

Beyond Haar measure

Haar measure isn't really the only reasonable way to pick a random matrix in $U \in \mathbb{G}$.

- Pick random M however you like, decompose $M = QR$, take $U = Q$.
- Pick independent reflections $M_1, M_2, \ldots, M_t$ across random hyperplanes (Householder transformations), take $U = M_1 M_2 \cdots M_t$.
- Heat kernel measure / Brownian motion on $\mathbb{G}$
- Pick M by Haar measure, take $U = M^k$ or $U = M^T M$.

We'll come back to (only) the last of these later on.

Additional references

- Kristopher Tapp, *Matrix Groups for Undergraduates*, 2nd edition, AMS, 2005.
- Francesco Mezzadri, "How to generate random matrices from the classical compact groups", *Notices of the AMS* 54, pp. 592–604, 2007.
- Benoît Collins and Piotr Śniady, "Integration with respect to the Haar measure on unitary, orthogonal and symplectic group", *Commun. Math. Phys.*, 264, pp. 773–795, 2006.
- Peter Forrester, *Log-Gases and Random Matrices*, Princeton, 2010.