Harmonic Analysis

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First lecture: A general frame for Fourier Analysis

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- Four different settings: $\mathbf{R^d}, \mathbf{T^d}, \mathbf{Z}, \mathbf{Z_N}$
- Why sines and cosines? The characters in a group, the Fourier transform.
- Translation invariant operators. Convolutions, multipliers, regularization, Young's inequality.
- Approximations of the identity. Density of test functions

Origins of Fourier Analysis

Solving the vibrating string, the heath diffusion problems with the separation of variable method and *the statement* that an arbitrary function in an interval can be expressed as a superposition of sines and cosines

$$f(x) = \sum_{n=0}^{+\infty} c_n(f) e^{2\pi i n x}$$

with

$$c_n(f) = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

A a- periodic function is superposition of the sines and cosines having period a. For (non-periodic) arbitrary functions, making $a \to +\infty$

$$f(x) = \int_{\mathbf{R}} c_\xi(f) e^{2\pi i x\cdot\xi} d\xi, c_\xi(f) = \int_{\mathbf{R}} f(x) e^{-2\pi i \xi\cdot x} dx.$$

Locally compact abelian groups

- Discrete: $\mathbf{Z^d}$, $Z_N = N$-th roots of unity.
- Continuous: $\mathbf{R^d}, \mathbf{T^d}$
- Haar Measure: Lebesgue $L^p(G)$ spaces
- Translation operator: $\tau_x, x \in G, \tau_x f)(y) = f(y-x)$.
- *Translation-invariant spaces*: spaces E such that $\tau_x f \in E$ whenever $f \in E$, and $\tau_x f$ is continuous in x. For instance, all L^p spaces are.
- $T : E \to E$ is said to be a *translation invariant operator (tip)* if $\tau_x(Tf) = T(\tau_x f), x \in G$.
- a differential operator T is time-invariant iff it has constant coefficients.

- We look for functions f such that the smallest translation-invariant space containing f has dimension one.
- This means that for $x \in G, \tau_x f$ must be a scalar multiple of f; $\tau_x f = \chi(-x)f, f(y-x) = \chi(-x)f(y), x, y \in G$.
- χ is continuous and so is f. Since $\tau_x \tau_y = \tau_{x+y}$, χ must satisfy $\chi(x+y) = \chi(x)\chi(y), x, y \in G$.
- Specializing to $y = 0$ yields $f(-x) = \chi(-x)f(0)$.
- Therefore f is a scalar multiple of χ.
- A *character* of G is a continuous non-zero homomorphism $\chi : G \to \mathbf{C}$.
- Bounded: $\chi : G \to \mathbf{T}$.
- The set of bounded characters of G has a natural group structure and constitute the so-called *dual group* $\widehat{G}$.

Why are characters useful?

- χ a character, T a translation invariant operator acting on a space containing χ,
- χ satisfies $\tau_x \chi = \chi(-x)\chi, x \in G$, as function of y
- If T commutes with translations

$$\tau_x(T\chi) = T(\tau_x \chi) = T(\chi(-x)\chi) = \chi(-x)T(\chi)$$

- $(T\chi)(y-x) = \chi(-x)T(\chi)(y), x, y \in G$.
- If we set $y = 0$, $T\chi = \lambda\chi$ with $\lambda = T(\chi)(0)$,
- *The characters of a group are eigenvectors of all translation invariant operators*

The group of characters in $\mathbf{R^d}$ and $\mathbf{T^d}$

- In $\mathbf{R}$, $\int_x^{x+h} \chi(z)dz = \int_0^h \chi(x+y)dy = \chi(x)\int_0^h \chi(y)dy$.
- This implies that χ is differentiable.

$$\chi'(x) = \lim_{y \to 0} \frac{\chi(x+y) - \chi(x)}{y} = \chi'(0)\chi(x).$$

- Hence $\chi(x) = e^{\alpha x}$ for some $\alpha \in \mathbf{C}$.
- In $\mathbf{R^d}, \alpha \in \mathbf{C^d}, \alpha \cdot \mathbf{x} = \alpha_\mathbf{1}\mathbf{x_1} + \cdots + \alpha_\mathbf{d}\mathbf{x_d}$.
- Bounded: $\alpha = 2\pi i \xi, e_\xi(x) = e^{2\pi i \xi \cdot x}, \xi \in \mathbf{R^d}$.
- In $\mathbf{T^d} = \mathbf{R^d}/\mathbf{Z^d}$: those of the above that are $\mathbf{Z^d}$-periodic, $\alpha = 2\pi i n$ for $n \in \mathbf{Z^d}$.
- Thus the dual group of $\mathbf{R^d}$ is identified with $\mathbf{R^d}$ and the dual group of $\mathbf{T^d}$ is identified with $\mathbf{Z^d}$.

The group of characters in $\mathbf{Z^d}$ and Z_N

- In $\mathbf{Z^d}, \mathbf{n} \mapsto \mathbf{z^n}, \mathbf{z} \in \mathbf{C^d}$, bounded characters correspond to $|z_j| = 1$,
$$\chi_z(n) = z^n = e^{2\pi i t \cdot n}.$$
Thus the dual group of $\mathbf{Z^d}$ is identified with $\mathbf{T^d}$.
- We identify the cyclic group $\mathbf{Z_N}$ with $\{0, 1, \ldots, N-1\}$ and functions there with N- periodic sequences $x = (x_n)$ indexed by $n \in \mathbf{Z}$.
- If $\omega_N = e^{2\pi i/N}$ denotes the primitive root of unity, it is immediate to check that the dual group is $\mathbf{Z_N}$ itself through
$$\psi_m(n) = \omega_N^{nm}, n \in \mathbf{Z}, \mathbf{m} = \mathbf{0}, \mathbf{1}, \ldots, \mathbf{N} - \mathbf{1}.$$

The Fourier transform

- In linear algebra, to deal with a linear operator T on $\mathbf{C^d}$ (look at vectors as functions defined on $1, 2, \dots n$) we try to diagonalize it in a basis of eigenvectors.
- Characters are eigenvectors of T.
- If the characters of G constitute a basis of some sort in E then we will have that *all translation-invariant operators on E diagonalize simultaneously in a basis of characters*.
- Complex exponentials are linearly independent

$$\sum_k c_k e^{2\pi i \xi_k \cdot x} = 0 \implies c_k = 0$$

- The Fourier transform of a function $f \in L^1(G)$ is the function $\hat{f}$ on $\widehat{G}$ defined by correlating with bounded characters

$$\hat{f}(\chi) = \langle f, \chi \rangle = \int_G f \overline{\chi} d\mu, f \in L^1(G)$$

- The map $f \in L^1(G) \mapsto \hat{f} \in L^\infty(\widehat{G})$ is called the Fourier transform in G.

Multiplier of a translation invariant operator

- $\widehat{G}$ has a natural structure of group and a Haar measure $d\nu$.
- Hope the χ behave like an orthonormal basis

$$f = \int_{\widehat{G}} \hat{f}(\chi)\chi \, d\nu(\chi) = \int_{\widehat{G}} \langle f, \chi \rangle \chi \, d\nu(\chi)$$

  that is (inversion formula)

$$f(x) = \int_{\widehat{G}} \hat{f}(\chi)\chi(x) \, d\nu(\chi), x \in G.$$

- Since T commutes with infinite sums and χ is a eigenvector of T, say $T(\chi) = m(\chi)\chi$,

$$Tf = \int_{\widehat{G}} \hat{f}(\chi) T(\chi) \, d\nu(\chi) = \int_{\widehat{G}} \hat{f}(\chi) m(\chi)\chi \, d\nu(\chi).$$

- $Tf(x) = \int_{\widehat{G}} \hat{f}(\chi) m(\chi)\chi(x) \, d\nu(\chi)$. m is the *multiplier* of T.

Another way to look at t.i.o.'s: convolution, impulse function

- Kernel of a linear operator $T : L^p(G) \to L^q(G)$

$$f = \int_G \delta_x f(x) d\mu(x) \implies Tf = \int_G T(\delta_x) f(x) d\mu(x),$$

$$Tf(y) = \int_G K_x(y) f(x) d\mu(x), K(x,y) = T(\delta_x)(y)$$

- $K(x,y)$ is the continuous version of a matrix.
- If T commutes with translations, set $g(y) = K_0(y) = T(\delta_0)(y)$. Then $\delta_x = \tau_x(\delta_0) \implies K_x = T(\delta_x) = T(\tau_x \delta_0) = \tau_x T(\delta_0) = \tau_x g, K(x,y) = g(y-x)$.
- Formally, all translations invariant operators are given by *convolution*

$$Tf(y) = (f * g)(y) = \int_G g(y-x) f(x) d\mu(x),$$

with a fixed "object" g, the *impulse response*.
- By now, g a function or a measure.

Relationship between both views

- There is a relation between g and m.
- This follows from the fact the Fourier transform of a convolution is the product of Fourier transforms

$$\widehat{f * g}(\chi) = \int_G (f * g)(y)\overline{\chi(y)}d\mu(y) =$$

$$= \int_G \int_G g(y-x)f(x)\overline{\chi(y-x)\chi(x)}d\mu(x)d\mu(y) =$$

$$= \hat{f}(\chi)\hat{g}(\chi)$$

- Hence, formally $m = \hat{g}$.

$$Tf = f * g \equiv \widehat{Tf} = m\hat{f}, m = \hat{g}.$$

Convolution as a mean

- $f * g$ is an (infinite) linear combination of translates of g,

$$f * g = \int_G \tau_x(g) f(x) d\mu(x).$$

- $f * g = g * f, f * (g * h) = (f * g) * h.$
- In case $g \in L^1(G), g \geq 0, \int_G g d\mu = 1,$

$$(f * g)(y) = \int_G f(y - x) g(x) d\mu(x) \text{weighted average of } f.$$

- $g = \frac{1}{\mu(B)} 1_B, (f * g)(y) =$ mean value of f in the ball $y + B$.
- Think g as density of a random variable X, then

$$f * g(y) = E(f(y - X)).$$

When is $f * g$ well defined? Properties of $f * g$

Schur's criteria for boundedness of operators $T : L^p(G) \to L^q(G)$

$$Tf(y) = \int_G K(x,y) f(x) d\mu(x).$$

- Assume $1 \leq p, q, r \leq +\infty$, $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$ and

$$\sup_x \left(\int_Y |K(x,y)|^r d\nu(y) \right)^{\frac{1}{r}} \leq C, \sup_y \left(\int_X |K(x,y)|^r d\mu(x) \right)^{\frac{1}{r}} \leq C.$$

- Then T is bounded from $L^p(X)$ to $L^p(Y)$ with constant $\|T\| \leq C$.
- $q = +\infty$ it follows from Holder's inequality (p, r conjugate exponents).
- $r = +\infty (p = q)$ is trivial: $|Tf| \leq C|f|$
- $p = +\infty (r = 1, q = +\infty$ trivial;
- $p = 1, r = q$ Continuous Minkowski inequality:

$$\| \int f(x) K(x, \cdot) d\mu(x) \|_q \leq \int |f(x)| \|K(x, \cdot)\|_q d\mu(x)$$

Proof of Schur's lemma

Assume all indexes are finite, positive $K, f \in L^p(X), g \in L^{q'}(Y)$. The hypothesis imply that

$$\frac{1}{r'} + \frac{1}{q} + \frac{1}{p'} = 1, \frac{p}{q} + \frac{p}{q'} = 1, \frac{r}{q} + \frac{r}{p'} = 1.$$

Using Holder's inequality with r', q, p',

$$|Tf(y)| \leq \int_X |f(x)|^{\frac{p}{r'}} |f(x)|^{\frac{p}{q}} |K(x,y)|^{\frac{r}{q}} |K(x,y)|^{\frac{r}{p'}} d\mu(x) \leq$$

$$\leq \|f\|_p^{\frac{p}{r'}} \left(\int_X |f(x)|^p |K(x,y)|^r d\mu(x) \right)^{\frac{1}{q}} \left(\int_X |K(x,y)|^r d\mu(x) \right)^{\frac{1}{p'}} \leq$$

$$\leq C^{\frac{r}{p'}} \|f\|_p^{\frac{p}{r'}} \left(\int_X |f(x)|^p |K(x,y)|^r d\mu(x) \right)^{\frac{1}{q}}$$

Continuation of Schur's proof. Young's inequality

Raising to q and integrating in y gives

$$\|Tf\|_q \leq C^{\frac{r}{p'}} \|f\|_p^{\frac{p}{r'}} \left(\int_X \int_Y |f(x)|^p |K(x,y)|^r d\mu(x) \right)^{\frac{1}{q}} =$$
$$= C^{\frac{r}{p'}} \|f\|_p^{\frac{p}{r'}} C^{\frac{r}{q}} \|f\|_p^{\frac{p}{q}} = C\|f\|_p$$

$K(x,y) = g(x-y) \to$ Young's inequality:

- Suppose $1 \leq p, q, r \leq +\infty, \frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}, f \in L^p(G), g \in L^r(G)$.
- Then
$$(f * g)(y) = \int_G g(y-x) f(x) d\mu(x),$$
converges absolutely for a.e y, $f * g \in L^q(G)$ and $\|f * g\|_q \leq \|f\|_p \|g\|_r$.

Local version of Young's inequality

In case $G = \mathbf{R^d}$ we can state a local version of Young's inequality, in which one of the functions has compact support while the other is locally in the corresponding L^p-space.

- Suppose $1 \leq p, q, r \leq +\infty, \frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}, f \in L^p_{loc}(G), g \in L^r_c(G)$.
- Then
$$f * g(y) = \int_G g(y-x)f(x)d\mu(x),$$
converges absolutely for a.e y, and $f * g \in L^q_{loc}(G)$.

Convolution as regularization

$G = \mathbf{R^d}, \mathbf{G} = \mathbf{T^d}$, look at the regularity properties of $f * g$.

- Suppose $1 \le p, r \le +\infty, \frac{1}{p} + \frac{1}{r} = 1$,
- Either $f \in L^p(G), g \in L^r(G)$, $f \in L^p_{loc}(G), g \in L^r_c(G)$ or $f \in L^p_c(G), g \in L^r_{loc}(G)$.
- Then $f * g$ is a continuous function.
- Assume that f (resp. g) is differentiable at every point and that its partial derivatives $\frac{\partial f}{\partial x_i}$ (respectively $\frac{\partial g}{\partial x_i}$) satisfy the same hypothesis of f (resp. g). Then $f * g$ is differentiable and

$$\frac{\partial}{\partial x_i}(f * g) = \frac{\partial f}{\partial x_i} * g, (\text{respectively} = f * \frac{\partial g}{\partial x_i}).$$

- $f * g$ inherits the regularity properties of both f, g.

$$D^\alpha(f * g) = (D^\alpha f) * g, (\text{respectively} = f * D^\alpha g),$$

holds whenever one the the right terms makes sense.

Approximate identities. Regularization

- Consider the group $\mathbf{Z}$. We will describe all translation invariant operators $T : l^1(\mathbf{Z}) \to \mathbf{l^q}(\mathbf{Z})$.
- Easy because the formal argument is OK: $\delta_0 \in l^1(Z)$. Define $g = T(\delta_0) = (g_n) \in l^q(\mathbf{Z})$.
- If $x = (x_n) \in l^1(\mathbf{Z}), \mathbf{x} = \sum \mathbf{x_n} \tau_\mathbf{n}(\delta_\mathbf{0})$ is convergent in $l^1(\mathbf{Z})$
- $Tx = \sum_n x_n \tau_n g, (Tx)_m = \sum_n x_n g_{m-n}, Tx = x * g$
- $T : l^1(\mathbf{Z}) \to \mathbf{l^q}(\mathbf{Z})\mathbf{t.i.o.} \equiv \mathbf{Tx} = \mathbf{x} * \mathbf{g}, \mathbf{g} \in \mathbf{l^q}(\mathbf{Z})$, by continuous Minkowsky inequality.

Approximations of the identity

- Non discrete groups $G = \mathbf{T^d}$ or $G = \mathbf{R^d}$, the delta mass is not a function but a measure, so it does not belong to any L^p space.
- However there is a good replacement for it. Note that δ is the formal unit for convolution, $f * \delta = f$.
- In what follows $G = \mathbf{T^d}$ or $G = \mathbf{R^d}$ with additive notation and dx Lebesgue measure.
- An approximate identity (or approximate kernel) is a family (k_ε) of functions in $L^1(G)$ satisfying
  1. $\int_G k_\varepsilon dx = 1$.
  2. $\int_G |k_\varepsilon| dx \leq C$, for some constant $C > 0$.
  3. For any $\delta > 0$, $\int_{|x|>\delta} |k_\varepsilon(x)| dx \to 0$ as $\varepsilon \to 0$.

Examples

- If $k \in L^1(G), \int_g k = 1$, set $k_\varepsilon(x) = \varepsilon^{-d} k(x/\varepsilon)$.
- The first two conditions are obvious, while for the third one

$$\int_{|x|>\delta} |k_\varepsilon(x)| dx = \int_{|x|>\frac{\delta}{\varepsilon}} |k(x)| dx \to 0.$$

  (rests of an absolutely convergent integral).
- The simplest example is to take as k the normalized characteristic function of the unit ball, $k(x) = \frac{1}{|B|}$ if $x \in B$ and zero otherwise.
- Then $f * k_\varepsilon(x)$ is simply the mean of f in $x + B_\varepsilon$, the ball centered at x of radious ε.

The Poisson and Gauss kernels

- The *Poisson family* in $\mathbf{R^d}$ that corresponds to

$$k(x) = c_d \frac{1}{(|x|^2 + 1)^{\frac{d+1}{2}}}, c_d = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}},$$

- The *Gaussian kernel* is given by

$$k(x) = \frac{1}{(\sqrt{2\pi})^d} e^{-\frac{1}{2}|x|^2}.$$

- On the torus $\mathbf{T^d}$ we will see later that natural examples appear when dealing with convergence of the Fourier series, namely the Fejer kernel. We may consider as well approximations of the identity indexed by $n \in \mathbf{N}$ with obvious modifications.

They do approximate δ_0

- If (k_ε) is an approximation of the identity and $f \in L^p(G), 1 \leq p < +\infty$, then $f * k_\varepsilon \to f$ in $L^p(G)$ as $\varepsilon \to 0$.
- If $f \in C_0(G)$, then $f * k_\varepsilon \to f$ uniformly on G.
- If $f \in L^1(G)$ and f is continuous at a point x_0, then $(f * k_\varepsilon)(x_o) \to f(x_0)$.

We can write

$$f - f * k_\varepsilon = f - \int_G \tau_x(f) k_\varepsilon(x) d\mu(x) = \int_G (f - \tau_x(f)) k_\varepsilon(x) d\mu(x),$$

and hence, by the continuous Minkowski inequality

$$\|f - f * k_\varepsilon\|_p \leq \int_G \|f - \tau_x(f))\|_p |k_\varepsilon(x)| d\mu(x).$$

Continuation of proof. Consequences

- To estimate it we break the above in two parts, corresponding to small x, say $\|x\| \leq \delta$, and $\|x\| > \delta$. The first one is estimated by

$$C \sup_{\|x\| \leq \delta} \|f - \tau_x(f)\|_p,$$

and hence can be made arbitrarily small if δ is small enough, uniformly in ε, due to the continuity of translations in $L^p(G)$, while the second is estimated by

$$2\|f\|_p \int_{|x|>\delta} |k_\varepsilon(x)| dx.$$

- A bounded operator T from $L^p(\mathbf{R^d})$ to $L^q(\mathbf{R^d})$, $1 \leq p, q < +\infty$ is a t.i.o. if and only if commutes with convolution with L^1 functions, that is,

$$T(f * g) = f * Tg, f \in L^1(\mathbf{R^d}), \mathbf{g} \in \mathbf{L^p}(\mathbf{R^d}).$$

- By Minkowski's continuous inequality, the right hand side of

$$f * g = \int_G (\tau_x g) f(x) d\mu(x),$$

is convergent in L^p, hence if T commutes with translations,

$$T(f*g) = \int_G f(x) T(\tau_x g)\, d\mu(x) = \int_G f(x) \tau_x Tg\, d\mu(x) = f*Tg.$$

- If T commutes with convolutions, we consider an approximation of the identity k_ε so that

$$\begin{aligned} T(\tau_x g) &= \lim_\varepsilon T((\tau_x g) * k_\varepsilon) = \lim_\varepsilon T(g * (\tau_x k_\varepsilon)) \\ &= \lim_\varepsilon (Tg) * (\tau_x k_\varepsilon) = \tau_x(\lim_\varepsilon (Tg) * k_\varepsilon) = \tau_x(Tg). \end{aligned}$$

Density of test functions

- The space of infinitely differentiable functions $C^\infty(\mathbf{T^d})$ is dense in all $L^p(\mathbf{T^d})$ spaces, $1 \leq p < \infty$. The space $C_c^\infty(\mathbf{R^d})$ of infinitely differentiable functions with compact support is dense in all $L^p(\mathbf{R^d})$ spaces, $1 \leq p < +\infty$.
- Proof: the space of continuous functions with compact support is dense. If f is in this space, and we take an approximation of the identity $k_\varepsilon(x) = \varepsilon^{-d} k(x/\varepsilon)$, with k a C^∞ function with compact support, then $k_\varepsilon * f \in C_c^\infty(\mathbf{R^d})$ and tends to f in L^p.
- The same proof shows that for an open set $U \subset \mathbf{R^d}$ the space $C_c^\infty(U)$ is dense in all $L^p(U)$ spaces as well.

Path to distributions

- If $f \in L^1_{loc}(U)$ and
$$\int_U f(x)\varphi(x)\,dx = 0$$
for all $\varphi \in C_c^\infty(U)$, then $f = 0$ a.e.
- The same is true if
$$\int_B f(x)dx = 0$$
for all balls $B \subset U$.
- *Remark*: for most of the approximations of the identity of type above, for $f \in L^1_{loc}(U)$, not only the means $f * k_\varepsilon \to f$ in $L^1_{loc}(U)$, but in fact we will see later that $f * k_\varepsilon \to f$ pointwise a.e. (Lebesgue theorem)

T.i.p's from $L^1(G)$

- The general form of a t.i.o. $T : L^1(\mathbf{R^d}) \to \mathbf{L^1}(\mathbf{R^d})$ is convolution with a finite complex Borel measure $d\mu$.
- Proof: Given such T, the idea is of course that $d\mu$ should be $T(\delta_0)$, we replace δ_0 by an approximation of the identity k_ε. Since they are bounded in L^1, $T(k_\varepsilon)$ will be also bounded in L^1. By the Banach-Alaoglu theorem there exists a finite complex valued measure $d\mu$ and a sequence $\varepsilon_n \to 0$ such that

$$\lim_n \int g(y) T(k_{\varepsilon_n})(y) dy = \int g(y) d\mu(y), g \in C_c.$$

Now, since $g = \lim_n g * k_{\varepsilon_n}$ and T is t.i.o. one has $Tg = \lim_n g * T(k_{\varepsilon_n})$. But

$$(g * Tk_{\varepsilon_n})(x) = \int g(x-y) T(k_{\varepsilon_n})(y) dy = \int g(x-y) d\mu(y) = (g * \mu)($$

- Hence T is convolution with μ on all functions with compact support and hence on all functions.

Second lecture: Fourier analysis in the torus

- The Fourier series of a periodic function.
- The Dirichlet, Fejer and Poisson means.
- Convergence in norm.
- Pointwise convergence.
- The rotation invariant operators in $L^1(T)$ and $L^2(T)$.
- The Fourier transform in $\mathbf{T^d}$

The Fourier series of a periodic function.

- Assume that the period is 1, we deal with functions on $\mathbf{T}$, parametrized by $e^{2\pi it}$.

$$(f * g)(t) = \int_{|t| \leq \frac{1}{2}} f(t-x)g(x)dx.$$

- Elementary blocks: $e_n(x) = e^{i2\pi nt}, n \in \mathbf{Z}$. The expression $\sum_n \langle f, e_n \rangle e_n$ is usually written

$$\sum_n c_n(f) e^{2\pi int},$$

with $c_n(f) = \int_0^1 f(t)e^{-2\pi int}\, dt, f \in L^1(\mathbf{T})$.

- $c_n(f)$ is called the n-th Fourier coefficient of f and the formal series

$$S(f) = \sum_{n \in \mathbf{Z}} c_n(f) e^{2\pi inx},$$

is called the *Fourier series* of f. Question: in what sense $f = S(f)$?

The Fourier basis

- The e_n constitute an orthonormal basis of $L^2(\mathbf{T})$: pairwise orthogonal,
$$\langle e_n, e_m \rangle = \int_0^1 e^{2\pi i(n-m)t} dt = 0$$
and their finite linear combinations are dense (Weierstrass theorem).
- This can be restated by saying that the map $f \to (c_n(f))_n$ is a bijection from $L^2(\mathbf{T})$ to $l^2(\mathbf{Z})$ satisfying
- $f(x) = \sum_n c_n(f) e^{2\pi inx}$ in $L^2(\mathbf{T})$
- $f(x) = \sum_n c_n e^{2\pi inx}, (c_n) \in l^2(\mathbf{Z})$, general expression.
- *Plancherel's identity*
$$\sum_n |c_n(f)|^2 = \int_0^1 |f(t)|^2 \, dt,$$
- polarized version *Parseval's relation*
$$\sum_n c_n(f) \overline{c_n(g)} = \int_0^1 f(t) \overline{g(t)} \, dt.$$

Properties of Fourier coefficients

- $c_n(f * g) = c_n(f)c_n(g)$.
- $c_n(\tau_x f) = e^{-2\pi i n x})c_n(f)$.
- If f is of class C^k and 2π- periodic, then $c_n(f^{(k)}) = (2\pi i n)^k c_n(f)$ and $c_n(f) = o(|n|^{-k})$.
- (The Riemann-Lebesgue lemma). $|c_n(f)| \leq \|f\|_1$ and $c_n(f) \to 0$ as $|n| \to \infty$.
- Proof: from the second it follows that

$$c_n(f - \tau_x f) = (1 - e^{-2\pi i n x} c_n(f)$$

  Choose $x = \frac{1}{2n} : 2|c_n(f)| \leq \|f - \tau_{2/n}\|_1$, so the result is a consequence of the continuity of translations.
- Nothing more general can be said regarding the speed of convergence.

The Dirichlet kernel

- To study $S(f)$ it is natural to consider the partial sums

$$S_N(f)(x) = \sum_{-N}^{N} c_n(f) e^{2\pi inx} =$$

$$= \int_0^1 f(t) \sum_{-N}^{N} e^{2\pi in(x-t)}\, dx = (f * D_N)(x),$$

$$D_N(x) = \sum_{n=-N}^{N} e^{2\pi int} = \frac{\sin(2N+1)\pi x}{\sin \pi x}.$$

- (D_N) is the *Dirichlet* kernel. It is NOT an approximation of the identity because $\|D_N\|$ behaves like $\log N$.
- However, we know $S_N f \to f$ in L^2 if $f \in L^2$.

The Fejer kernel

- $\sigma_N(f) = \frac{1}{N+1}(S_0(f) + \cdots + S_N(f)) = (f * \sigma_N)$,

$$\sigma_N(x) = \frac{1}{N+1}(D_0(x) + \cdots + D_N(x)) = \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) e^{2\pi inx}$$

$$= \frac{1}{N+1}\left(\frac{\sin(N+1)\pi x}{\sin \pi x}\right)^2$$

- σ_N is positive, has integral one and if $|x| > \delta$ then $|\sigma_N(x)| \leq c_\delta \frac{1}{N}$, so it is an approximation of the identity.
- If $f \in L^p(\mathbf{T}), \mathbf{p} < +\infty$. then $\sigma_N(f) \to f$ in $L^p(\mathbf{T})$.
- If f is continuous at t_0, then $\sigma_N(f)(t_0) \to f(t_0)$.
- If f is continuous then $\sigma_N(f) \to f$ uniformly (Weierstrass thm)
- Uniqueness theorem: If $c_n(f) = 0$ for all n then $f = 0$.
- If f is continuous at x_0 and $S(f)(x_0)$ converges, sum must be $f(t_0)$

The Poisson kernel

- Dirichlet problem in the unit disc **D**, that is, given $f \in C(\mathbf{T})$ to find u s.t. $\Delta u = 0, u \in C(\overline{D}), u = f \text{ on} T$.
- We solve this problem exploiting its invariance by rotations.
- For each $0 < r < 1$ we pose $u_r(t) = u(re^{2\pi it})$; the operator $f \to u_r$ is rotation invariant (because Δ is) hence it must be given by a circular convolution with some $P_r, u_r = f * P_r$,
- and must diagonalize in the Fourier basis, i.e. $c_n(u_r) = m_n c_n(f), m_n = c_n(P_r)$.
- the solution of the Dirichlet problem for $f(t) = e^{2\pi inx}$ is $u(re^{2\pi it}) = z^n$ if n is positive and $\overline{z}^n$ if n is negative, $z = re^{2\pi it}$, so $m_n = r^{|n|}$.
- Since

$$P_r(x) = \sum_n r^{|n|} e^{2\pi inx} = \frac{1 - r^2}{|1 - re^{2\pi ix}|^2} = \frac{1 - r^2}{1 + r^2 - 2r\cos(2\pi x)},$$

  we guess that the solution should be

- $u(re^{2\pi it}) = (f * P_r)(t) = \int_0^1 f(x) \frac{1-r^2}{1+r^2-2r\cos(2\pi(t-x))} dx.$
- The kernel P_r is called the *Poisson kernel*. It is positive with integral one; if $|t| > \delta$, then $|1 - re^{2\pi it}| \geq c - \delta$, whence

$$P_r(t) \leq c_\delta(1 - r^2).$$

  -(P_r) is an approximation of the identity as $r \to 1$.
- For $f \in L^1(\mathbf{T})$, the function u defined on the unit disc by ( ) above is an harmonic function in D satisfying $u_r \to f$ in $L^1(\mathbf{T})$. In case $f \in C(T)$, u is continuous in the closed disc with boundary values equal to f and is thus the solution of Dirichlet's problem.

Pointwise convergence

- Pointwise convergence of the Fourier series of f: when $f(x) = \sum_n c_n(f) e^{2\pi inx}$ a.e. or at a given point?
- Using sumability "a la Fejer" o " a la Poisson" the situation is quite good. Indeed, as both the Fejer and Poisson kernels are approximate identities one can prove that for $f \in L^1(\mathbf{T})$ both $F_N(f)(t)$ and $u_r(t)$ have limit $f(t)$ a.e. We will see this later as an application of the maximal function of Hardy-Littlewood.
- The a.e. pointwise convergence of $S_N(f)(x)$ to $f(x)$ is a extremely hard question.
- Kolmogorov constructed an $f \in L^1(\mathbf{T})$ such that $S(f)$ diverges a.e.

- The poinwise convergence of $S(f)$ for $f \in L^p(\mathbf{T})$ was a very hard open problem in Fourier analysis till Carleson proved that $S(f)(t)$ converges to $f(t)$ for a.e. t for $f \in L^2(T)$ in a celebrated breakthrough, and this was generalized to $L^p(\mathbf{T}), \mathbf{1} < \mathbf{p} < +\infty$ by Hunt.
- Convergence of $S(f)$ at a point x_0 where f is continuous is not guaranteed. But if the modulus of continuity of f at x_0 satisfies a Dini type condition then $S(f)(x_0)$ converges to $f(x_0)$.
- In particular this is the case if f satisfies a Lipschitz condition or if it is differentiable at x_0.

Convergence in norm

- Regarding convergence in $L^p(\mathbf{T}), \mathbf{1} \leq \mathbf{p} < +\infty$, we have seen that $\sigma_N(f)$ and u_r have limit f in $L^p(\mathbf{T})$ while we trivially know that $S_N(f) \to f$ in $L^2(\mathbf{T})$ if $f \in L^2(\mathbf{T})$.
- We will see later, as an application of the CZ theory, that this holds true for $1 < p < +\infty$.
- Uniform convergence of $S(f)$ when f is continuous.
- du Bois-Reymond constructed a continuous f such that $S(f)$ diverges at some point (in fact examples can be constructed where $S(f)$ diverges on a dense set).
- Sufficient conditions can be given. For instance, if f is continuous and of bounded variation then $S(f)$ converges to f uniformly.

Rotation invariant operators in $L^p(\mathbf{T})$

For a bounded operator $T : L^p(\mathbf{T}) \to \mathbf{L^p(T), 1 \leq p}$, the following are equivalent:

- It commutes with rotations.
- It commutes with convolution with $L^1(\mathbf{T})$ functions.
- It diagonalizes in the Fourier basis: $c_n(Tf) = m_n c_n(f), n \in \mathbf{Z}$.

Moreover, the general form of T is given

- In case $p = 1$, by $Tf = f * \mu$, with a finite complex Borel measure μ, in which case $m_n = c_n(\mu)$.
- In case $p = 2$, by $c_n(Tf) = m_n c_n(f)$ with m_n an arbitrary bounded sequence, in which case the norm of T as an operator in $L^2(\mathbf{T})$ equals $\sup_n |m_n|$.
- In both cases T is convolution with $g = \sum_n m_m e^{2\pi inx}$.
- $p = 1$: g is a measure, but not able to describe the m_n.
- $p = 2$: not able to describe g, describe exactly its coefficients.
- Note that Young inequalities would not prove the result for $p = 2$ because $g \notin L^1(\mathbf{T})$.

The Fourier transform in $\mathbf{T^d}$

- If f is a-periodic in $\mathbf{R}$ the Fourier series becomes

$$f(x) = \sum_n c_n(f) e^{2\pi i \frac{n}{a} x}, c_n(f) = \frac{1}{a} \int_0^a f(x) e^{-2\pi i \frac{n}{a} x}\, dx.$$

  Frequencies located at the integer multiples of $\frac{1}{a}$.

- The Fourier series of a function on $\mathbf{T^d}$ is

$$Sf(t) = \sum_{k \in \mathbf{Z^d}} c_k(f) e^{2\pi i k \cdot t}, k \cdot t = k_1 t_1 + \cdots + k_d t_d,$$

  with

$$c_k(f) = \int_0^1 \ldots \int_0^1 f(t) e^{-2\pi i k \cdot t}\, dt_1 \ldots dt_d.$$

  g non trivial periodic function in $\mathbf{R^d}, \mathbf{d > 1}$; its group of periods is a lattice $\Lambda = A(\mathbf{Z^d}), \mathbf{A} \in \mathbf{GL(d)}$ with fundamental region $I = A([0,1]^n)$.

If $f(t) = g(At)$, f is $\mathbf{Z^d}$ periodic and has a Fourier series expansion Rewriting it in terms of g one obtains, with

$$\Lambda^* = (A^*)^{-1}(\mathbf{Z^d})$$

being the dual lattice

$$g(t) = \sum_{\rho \in \Lambda^*} c_\rho(g) e^{2\pi i \rho \cdot t},$$

where

$$c_\rho = \frac{1}{|detA|} \int_I g(t) e^{-2\pi i \rho \cdot t} dt.$$

The frequencies are then located at Λ^*.

- Much of the analysis done in the previous section goes over to $N > 1$, provided that appropriate definitions are given, namely that of $S_N f(t)$.
- If rectangulars sums are used, that is,

$$S_N^r(f)(t) = \sum_{|k_i| \leq N} c_k(f) e^{2\pi i k \cdot t},$$

  and correspondingly for $\sigma_N(f)$, then the results for $S_N(f)$ and $\sigma_N(f)$ hold as well.
- However, if spherical sums are considered

$$S_N^e(f)(t) = \sum_{|k| \leq N} c_k(f) e^{2\pi i k \cdot t}, |k|^2 = k_1^2 + \cdots + k_d^2,$$

  then the situation becomes more complicated.

Third lecture: Fourier analysis in $\mathbf{R^d}$

- The Fourier transform in $L^1(\mathbf{R^d})$, uniqueness theorem, the inversion formula
- The Fourier transform in $L^2(\mathbf{R^d})$.
- Translation invariant operators in $L^1(\mathbf{R^d})$ and $L^2(\mathbf{R^d})$.

The Fourier transform in $L^1(\mathbf{R^d})$

$\hat{f}(\xi) = \langle f, e_\xi \rangle = \int_{\mathbf{R^d}} f(x) e^{-2\pi i \xi \cdot x}\, dx, \xi \in \mathbf{R^d}, \mathbf{f} \in \mathbf{L^1}(\mathbf{R^d})$
$\hat{\mu}(\xi) = \int_{\mathbf{R^d}} e^{-2\pi i \xi \cdot x}\, dx., \mu$measure.

- $\widehat{\tau_x f}(\xi) = e^{-2\pi i \xi \cdot x} \hat{f}(\xi)$
- If $g(x) = e^{2\pi i \eta \cdot x} f(x)$, then $\hat{g}(\xi) = \tau_\eta(\xi)$.
- $\widehat{D_\lambda f}(\xi) = \lambda^{-d} \hat{f}(\frac{\xi}{\lambda})$.
- $\widehat{D^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$.
- $(D^\alpha \hat{f})(\xi) = ((-2\pi i x)^\alpha f(x))\widehat{}(\xi)$.
- $\widehat{f * g} = \hat{f} \hat{g}$.
- If A is an invertible matrix and $f_A(x) = f(Ax)$, then

$$\widehat{f_A}(\xi) = \frac{1}{|detA|} \hat{f}(A^{-1})^* \xi).$$

- (The Riemann-Lebesgue lemma). $\hat{f}$ is a continuous function vanishing at ∞.

- Fourier transform commutes with composition with orthogonal matrices A.
- $f(x)$ is radial $\implies \hat{f}$ radial.
- f, g radial $\implies f * g$ radial.
- $P(D) = \sum_{\alpha \in \mathbf{N^d}} c_\alpha D^\alpha$ is a differential operator with constant coefficients (and so translation invariant)

$$\widehat{P(D)f}(\xi) = P(2\pi i \xi)\hat{f}(\xi), (P(D)\hat{f})(\xi) = (P(-2\pi i x)f)\widehat{\ }(\xi),$$

- A translation- invariant operator T has a multiplier, for instance that of $P(D)$ is $m(\xi) = P(2\pi i \xi)$.

- We say that T is invariant by rigid motions if moreover $T(f_A) = (Tf)_A$ for all orthogonal matrices. Then, its multiplier must be radial. For instance, the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2},$$

is radial and has multiplier $m(\xi) = -4\pi^2|\xi|^2$.
- If a differential operator $P(D)$ is invariant by rigid motions, then its multiplier is a radial polynomial, that is a polynomial in $|\xi|^2$, and hence we have
- A differential operator $P(D)$ is invariant by rigid motions if and only if it is a polynomial in Δ.

The Dirichlet, Fejer and Poisson kernels in $\mathbf{R^d}$

$f = \int_{\mathbf{R^d}} \langle f, e_\xi \rangle e_\xi d\xi, f(x) = \int_{\mathbf{R^d}} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi,$

- Dirichlet means with cubes or balls. Quite different behavior.

$$(S_R^c f)(x) = \int_{|\xi_i|\leq R} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi$$

$$(S_R^b f)(x) = \int_{|\xi|\leq R} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi$$

$$(S_R^c f)(x) = (f * D_R)(x), D_R(x) = \prod_{j=1}^{d} \frac{\sin 2\pi R x_j}{\pi x_j}.$$

- D_R is strictly speaking not integrable (a typical example of a conditionally convergent integral) and is NOT an approximation of the identity.

But their means are an approximation of the identity

$$(\sigma_R f)(x) = \frac{1}{R}\int_0^R (S_r^c f)(x)\,dr = (f * F_R)(x),$$

$F_R(x) = R^d F(Rx)$,

$$F(x) = \prod_{i=1}^{d} \frac{1-\cos 2\pi x_j}{2\pi^2 x_j^2}.$$

F has integral one and hence F_R is an approximation of the identity.

Poisson and Gauss means

- The general scheme: continuous integrable function $\Phi, \Phi(0) = 1$

$$f_\Phi(\varepsilon, x) = \int_{\mathbf{R^d}} \hat{f}(\xi) e^{2\pi i x\cdot\xi} \Phi(\varepsilon\xi) d\xi.$$

- Fubini's theorem implies that

$$\int_{\mathbf{R^d}} \hat{f}(\xi) g(\xi) d\xi = \int_{\mathbf{R^d}} f(y)\hat{g}(y) dy, f, g \in L^1(\mathbf{R^d}).$$

- The Fourier transform of $\Phi(\varepsilon\xi)$ is $(\varepsilon)^{-d}\widehat{\Phi}(\frac{y}{\varepsilon}) = \hat{\Phi}_\varepsilon(y)$, whence the Fourier transform of $e^{2\pi i x\cdot\xi}\Phi(\varepsilon\xi)$ is $\hat{\Phi}_\varepsilon(y - x)$ and so

$$f_\Phi(\varepsilon, x) = (f * \widehat{\Phi}_\varepsilon)(x).$$

- Choose Φ so that $\int \widehat{\Phi} = 1 \to$ approximation of the identity.

Heath diffusion

- Choose $\Phi(x) = e^{-\pi|x|^2}, \widehat{\Phi}(\xi) = e^{-\pi|\xi|^2}, \widehat{\Phi} = \Phi$.
  $f_\Phi(\varepsilon, x) = \int_{\mathbf{R^d}} \hat{f}(\xi) e^{-\pi\varepsilon^2|\xi|^2} e^{2\pi i x\cdot\xi}\, d\xi = (f * \Phi_\varepsilon)(x)$,
- But $\int_{\mathbf{R^d}} \Phi(x)dx = \widehat{\Phi}(0) = \Phi(0) = 1$, therefore Φ_ε is an approximation of the identity so $f_\phi(\varepsilon, x) \to f(x)$ in $L^1(\mathbf{R^d})$.
- Unicity theorem: if $\widehat{f} = 0$, then $f = 0$. It also implies
- Inversion theorem: If $f \in L^1(\mathbf{R^d})$ and $f \in L^1(\mathbf{R^d})$ then

$$f(x) = \int_{\mathbf{R^d}} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi, a.e.x$$

  and in particular f is a.e. equal to a continuous function vanishing at infinity.
- If we put $\mathcal{F}$ for the Fourier transform, this means $\mathcal{F}^{-1}f(x) = \mathcal{F}f(-x)$, that we call the *Fourier cotransform*.
- Connection with heath diffusion: $u(t, x) = f_\phi(\sqrt{t}, x)$ is the solution of the heath equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\Delta_x u(x, t), u(0, x) = f(x).$$

Harmonic extension

- More generally, choose any continuous function $\Phi \in L^1$ such that $\Phi(0) = 1$ and $\widehat{\Phi}$ is integrable. Then by the above the integral of $\widehat{\Phi}$ equals $\Phi(0) = 1$ and we can repeat the same argument.
- Another choice is $\Phi(x) = e^{-2\pi|x|}$ (Abel means). In this case

$$\widehat{\Phi}(\xi) = c_d \frac{1}{(1 + |\xi|^2)^{(d+1)/2}}, c_d = \frac{\Gamma[\frac{d+1}{2}]}{\pi^{(d+1)/2}}.$$

- In this case $u(t, x) = f_\phi(t, x)$ satisfies

$$\frac{\partial^2 u}{\partial t^2} + \Delta_x u(t, x) = 0, u(0, x) = f(x),$$

that is, is the solution of the Dirichlet problem in the half-space.

The Fourier transform in $L^2(\mathbf{R^d})$

- Schwarz class $\mathcal{S}(\mathbf{R^d})$ of C^∞ functions f such that
$$\lim_{|x|\to+\infty} |x^\beta D^\alpha f(x)| = 0, \alpha, \beta \in \mathbf{N^d}.$$
Dense in all L^p spaces, $1 \leq p < +\infty$, contains $C_c^\infty(\mathbf{R^d})$.
- The Fourier transform is a bijection from $\mathcal{S}(\mathbf{R^d})$ to itself that transforms convolutions into multiplication and conversely.
- Applying $\int f\hat{g} = \int \hat{f}g$ to $f \in \mathcal{S}$, we find
$$\|f\|_2 = \|\hat{f}\|_2, f \in \mathcal{S}$$
- Thus we have that the Fourier transform is an isometry between $\mathcal{S}$ and itself, so extends to an isometry of the whole of L^2,
$$\hat{f}(\xi) = \lim_{R\to+\infty} \int_{|x|\leq R} f(x)e^{-2\pi i\xi\cdot x}\,dx,$$
exists in $L^2(\mathbf{R^d})$, defines $\hat{f}$, and
$$\|\hat{f}\|_2 = \|f\|_2, Plancherel's identity$$

A miracle?

$$f = \int_{\mathbf{R^d}} \langle f, e_\xi \rangle e_\xi d\xi.$$

- The $e_\xi \notin L^2(\mathbf{R^d})$ yet they behave as if they were an orthonormal basis of $L^2(\mathbf{R^d})$. True bases: the wavelet bases.
- Kernel $K(x,\xi) = e^{2\pi i x \cdot \xi}$ of modulus one; ensures $L^1(\mathbf{R^d})$ to $L^\infty(\mathbf{R^d})$, but its boundedness in L^2 depends on much more than size, depends on cancelations.

$$\int_{\mathbf{R^d}} |\hat{f}(\xi)|^2 \, d\xi = \int_{\mathbf{R^d}} \int_{\mathbf{R^d}} \int_{\mathbf{R^d}} f(x)\overline{f(y)} e^{2\pi i \xi (y-x)} dx dy d\xi.$$

- This being equal to $\int_{\mathbf{R^d}} |f(x)|^2 \, dx$ means formally that

$$\int_{\mathbf{R^d}} e^{2\pi i \xi x} \, d\xi = \delta_0(x).$$

- The above says that superposition of all frequencies is zero outside zero.

Translation invariant operators in $L^p(\mathbf{R^d})$

- For a bounded operator $T : L^p(\mathbf{R^d}) \to \mathbf{L^p(R^d), p = 1, 2}$, the following are equivalent:
  1. It commutes with translations.
  2. It commutes with convolution with $L^1(\mathbf{R^d})$ functions.
  3. It diagonalizes in the Fourier basis: $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$.
- Moreover, the general form of T is given by
  1. in case $p = 1$, $Tf = f * \mu$, with a finite complex Borel measure μ, in which case $m(\xi) = \hat{\mu}(\xi)$.
  2. in case $p = 2$, $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$ with m an arbitrary bounded function.
- The t.i.p on $\mathcal{S}(\mathbf{R^d})$, by Fourier transform correspond to multiplication operators acting on $\mathcal{S}(\mathbf{R^d})$, $\widehat{Tf} = m\hat{f}$, with m a $C^\infty(\mathbf{R^d})$ of *slow growth* meaning that for every $\alpha \in \mathbf{N^d}$ there exists $k \in \mathbf{N}$ such that $|D^{alpha} m(x) = O(|x|^k)$.
- Do not know exactly m when $p = 1$, we do not know exactly what is

$$\int_{\mathbf{R^d}} m(\xi) e^{2\pi i x \cdot \xi} d\xi, m \in L^\infty(\mathbf{R^d}).$$

Translation invariant subspaces of $L^2(\mathbf{R^d})$

- The last result serves to describe all closed translations invariant subspaces E of $L^2(\mathbf{R^d})$.
- Associate to E the projection operator P onto E, that is, $Pf \in E$ and $f - Pf$ is orthogonal to E, $P^2 = P$.
- If E is invariant by translations so is P, hence it has a bounded multiplier $m \in Ł^\infty(\mathbf{R^d})$.
- Now, $P^2 = P$ translates to $m^2 = m$, whence $m = 0$ or $m = 1$.
- Let A be the set where $m = 0$. A given $f \in E$ if and only if $Pf = f$, that is $m\hat{f} = \hat{f}$, whence it follows that $\dot{f} \in E$ if and only if $\hat{f}$ vanishes a.e. on A.
- This is the general form of a closed translation invariant subspace in $L^2(\mathbf{R^d})$. In particular, the translates of a a given function $f \in L^2(\mathbf{R^d})$ span the whole of $L^2(\mathbf{R^d})$ if and only if $\hat{f} \neq 0$ a.e. (Beurling's theorem)

Fourth lecture: Distributions in Harmonic Analysis

- The notion of distribution. Operations with distributions.
- Convergence of distributions
- Distributions with compact support and tempered distributions.
- Fourier transform of tempered distributions.
- Convolution of distributions
- Translation invariant operators in test spaces and in spaces of distributions.
- Fundamental solutions of linear constant coefficient PDE's.
- Poisson's summation formula. An unified language

What is a distribution in an open set $U \subset \mathbf{R^d}$?

- Basic idea is to consider that functions f are not given by their values at points but by their action on other functions by integration.
- $\mathcal{D}(\mathbf{U})$ dense in all $L^p(U)$ spaces. Hence, if $f, g \in L^1_{loc}(U)$

$$\int_U f(x)\varphi(x)dx = \int_U g(x)\varphi(x)dx, \forall \varphi \in \mathcal{D}(\mathbf{U}) \implies \mathbf{f} = \mathbf{g} \ \mathbf{a}.\mathbf{e}.$$

- This means that f is completely known as soon as one knows

$$u_f(\varphi) = \int_U f(x)\varphi(x)\, dx,$$

- A *distribution* on U is a continuous linear map $u : \mathcal{D}(\mathbf{U}) \to \mathbf{C}$.
- Continuity: if $\varphi_n \in$ tends to zero (meaning that they have their supports in a fixed compact set K of U and $D^\alpha(\varphi_n) \to 0$ uniformly in K for all α), then $u(\varphi_n) \to 0$.
- It is customary to write $u(\varphi) = \langle u, \varphi \rangle$. The space of distributions on U is denoted $\mathcal{D}'(\mathcal{U})$

Examples

- Functions $f \in L^1_{loc}$ are distributions.
- A locally finite measure $d\nu$ on U is also a distribution.
- The Dirac measure at a will be denoted δ_a.
- If Λ is a discrete set in U (hence countable), the comb

$$\sum_{a \in \Lambda} \delta_a,$$

  is also a distribution.
- Example of a distribution that is not a function nor a measure.

$$\langle p.v.\frac{1}{x}, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{\varphi(x)}{x}\, dx.$$

  Note that the limit exists because it equals

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{\varphi(x) - \varphi(-x)}{x}\, dx.$$

Operations with distributions

- When defining an operation on distributions we look for consistency
- Definition of $\tau_x u$ should be so that $\tau_x u_f = u_{\tau_x f}$ for $f \in L^1_{loc}$.

$$\int_{\mathbf{R^d}} \tau_x f(y)\varphi(y)dy = \int_{\mathbf{R^d}} f(y-x)\varphi(y)dy =$$
$$= \int_{\mathbf{R^d}} f(z)\varphi(z+x)dz = \int_{\mathbf{R^d}} f(z)\tau_{-x}\varphi(z)\,dz,$$

- Must define
$$\langle \tau_x u, \varphi\rangle = \langle u, \tau_{-x}\varphi\rangle.$$
- $u \in \mathcal{D}'(\mathbf{R^d})$ *a- periodic* if $\tau_a u = u$. All a-periodic functions are, and also the Dirac comb $\Delta_a = \sum_{n\in\mathbf{Z}} \delta_{na}$.
- Product of $u \in \mathcal{D}'(\mathcal{U})$ with $g \in C^\infty(U)$: $\langle gu, \varphi\rangle = \langle u, g\varphi\rangle$. $g\delta_a = g(a)\delta_a, g\Delta_a = \sum_n g(na)\delta_{na}, x\, v.p.\frac{1}{x} = 1.$

Derivatives of distributions

- Derivative $D^\alpha u : \langle D^\alpha u, \varphi\rangle = (-1)^{|\alpha|}\langle u, D^\alpha \varphi\rangle$.
  In $\mathbf{R}$, $\int f'\varphi = -\int f(\varphi)'$ holds for all locally absolutely continuous functions (undefinite integrals of integrable functions), so that $(u_f)' = u_{f'}$ for those.
- Unit step of Heaviside function H, 1 for positive x and zero for negative x. Then $H' = \delta_0$ because:

$$\langle H', \varphi\rangle = -\langle H, \varphi'\rangle = -\int_0^\infty \varphi'(x)dx = \varphi(0),$$

- A function f which is continuously differentiable in the closed intervals determined by some points $a_1, \ldots, a_N$ where it has some jump discontinuities with jumps s_i. Then $(u_f)' = u_{f'} + \sum_i s_i \delta_{a_i}$.
- The a- periodic function which in each interval $[na, (n+1)a]$ is linear from 0 to 1 has derivative $\frac{1}{a} - \Delta_a$.

- Derivative of $\log|x|$ is $p.v.\frac{1}{x}$.

$$-\int_{\mathbf{R}} \log|x|\varphi'(x)\,dx = -\lim_{\varepsilon}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty}\right)\log|x|\varphi'(x)\,dx$$

$$\lim_{\varepsilon}(\varphi(\varepsilon)-\varphi(-\varepsilon))\log\varepsilon + \lim_{\varepsilon}\int_{|x|>\varepsilon}\frac{\varphi(x)}{x}\,dx,$$

- One can prove, in **R**, that if $u' = 0$ then u is constant and that every distribution has a primitive.

Convergence of distributions

- $u_n \to u$ means simply $\langle u_n, \varphi \rangle \to 0$ for all φ.
- With this definition all operations are continuous, in particular the differentiation.
- In particular we can consider series of distributions. We will be interested in trigonometric series

$$\sum_{n \in \mathbf{Z}} c_n e^{2\pi i \frac{n}{a} x}.$$

- Partial sums act as

$$\langle \sum_{n=-N}^{N} c_n e^{2\pi i \frac{n}{a} x}, \varphi \rangle = \sum_{n=-N}^{N} c_n \hat{\varphi}(-\frac{n}{a}).$$

  $\hat{\varphi} \in \mathcal{S}(\mathbf{R^d})$, $\hat{\varphi}(-\frac{n}{a}) = O(|n|^{-k})$ for all k.
- If $c_n = O(|n|^k)$ for some k then the series indeed defines a distribution. This is not necessarily the Fourier series of a periodic function.

Distributions with compact support

- A distribution with compact support is a continuous linear map $T : C^\infty(\mathbf{R^d}) \to \mathbf{C}$
- Continuity means here that if $\varphi_n \in C^\infty(\mathbf{R^d})$ tend to zero (meaning that $D^\alpha \varphi_n(x) \to 0$ uniformly on compacts) then $\langle T, \varphi_n \rangle \to 0$.
- Again, we may think that T has compact support if it is capable to act against all $C^\infty(\mathbf{R^d})$ functions. The space of distributions with compact support is denoted $\mathcal{E}'(\mathbf{R^d})$

- Consider the a- periodic function f equal to $\frac{x}{a}$ in $[0, a]$. By direct computation

$$f(x) = \frac{1}{2} + \frac{i}{2\pi} \sum_{n \neq 0} \frac{1}{n} e^{2\pi i \frac{n}{a} x}.$$

- Convergent in $L^2(\mathbf{T}) \implies$ convergent as distributions,

$$f'(x) = -\frac{1}{a} \sum_{n \neq 0} e^{2\pi i \frac{n}{a} x}.$$

- Shaw before that $f' = \frac{1}{a} - \Delta_a$, so

$$\Delta_a = \sum_{n \in \mathbf{Z}} \delta_{na} = \frac{1}{a} \sum_{n \in \mathbf{Z}} e^{2\pi i \frac{n}{a} x}.$$

Tempered distributions

- Would like to define the Fourier transform of a distribution.
- For $f \in L^1(\mathbf{R^d})$, $\int_{\mathbf{R^d}} \hat{\mathbf{f}}(\mathbf{x})\varphi(\mathbf{x})\mathbf{dx} = \int_{\mathbf{R^d}} \mathbf{f}(\mathbf{x})\hat{\varphi}(\mathbf{x})\mathbf{dx}$, so should define $\langle \hat{u}, \varphi \rangle = \langle u, \hat{\varphi} \rangle$.
- Problem is that $\hat{\varphi}$ is no longer in $\mathcal{D}(\mathbf{R^d})$. Must restrict to a particular class of distributions. The Schwarz space is invariant by the Fourier transform, so the above would work if $\mathcal{S}(\mathbf{R^d})$ were used instead of $\mathcal{D}(\mathbf{R^d})$
- A tempered distribution is a continuous linear map $u : \mathcal{S}(\mathbf{R^d}) \to \mathbf{C}$
- Continuity means: $\varphi_n \to 0$ in $\mathcal{S}(\mathbf{R^d})$ (meaning that $\sup_x |x|^{|\beta|} |D^\alpha \varphi_n(x)| \to 0$ as $n \to +\infty$ for all $\alpha, \beta \in \mathbf{N^d}$) then $\langle u, \varphi_n \rangle \to 0$.
- The restriction of u to $\mathcal{D}(\mathbf{R^d})$ is then a distribution (and in fact u is completely determined by this restriction since $\mathcal{D}(\mathbf{R^d})$ is dense in $\mathcal{S}(\mathbf{R^d})$).

Examples

- Tempered distributions as those capable to act on $\mathcal{S}(\mathbf{R^d})$. Denote by $\mathcal{S}'(\mathbf{R^d})$ the space of tempered distributions.
- All distributions with compact support are in $\mathcal{S}'(\mathbf{R^d})$.
- Among the $f \in L^1_{loc}$, those with slow growth, meaning that $|f(x)| = O(|x|^k)$ for some integer k are in $\mathcal{S}'(\mathbf{R^d})$.
- All L^p- functions, $1 \leq p \leq +\infty$ are as well.
- All L^1_{loc} periodic functions are in $\mathcal{S}'(\mathbf{R^d})$.
- Easy to see that $gu \in \mathcal{S}'(\mathbf{R^d})$ if $u \in \mathcal{S}'(\mathbf{R^d})$ and $g \in C^\infty(\mathbf{R^d})$ and its derivatives have slow growth, that is, for all $\alpha \in \mathbf{N^d}, |\mathbf{D}^\alpha \mathbf{g}(\mathbf{x})| = \mathbf{O}(|\mathbf{x}|^\mathbf{k})$ for some k because in this case $g\varphi \in \mathcal{S}(\mathbf{R^d})$ for all $\varphi \in \mathcal{S}(\mathbf{R^d})$.
- We denote by $\mathcal{B}(\mathbf{R^d})$ the class of these g

Fourier transform of tempered distributions

- The Fourier transform $\hat{T}$ of a tempered distribution is thus defined by $\langle \hat{u}, \varphi \rangle = \langle u, \hat{\varphi} \rangle$.
- Since the Fourier transform in $\mathcal{S}(\mathbf{R^d})$ is an isomorphism the same happens with $\mathcal{S}'(\mathbf{R^d})$.
- Properties of the Fourier transform regarding translations vs multiplication by exponentials and derivatives vs multiplication by polynomials go over to $\mathcal{S}'(\mathbf{R^d})$.

Examples

- $\hat{\delta}_a(\xi) = -e^{2\pi i a\xi}, \widehat{e^{2\pi iax}} = \delta_a$. In particular, $\hat{\delta}_0 = 1, \hat{1} = \delta_0$.
- In particular $\hat{\Delta}_a = \sum_n \widehat{\delta_{na}} = \sum_n e^{2\pi i n a\xi} = \frac{1}{a}\Delta_{\frac{1}{a}}$.
- In particular, Δ_1 is its own Fourier transform.
- Let us compute the Fourier transform of $p.v.\frac{1}{x}$. Its action on φ is

$$\lim_{\varepsilon} \int_{\varepsilon<|xi|<1/\varepsilon} \frac{\hat{\varphi}(\xi)}{\xi}\, d\xi =$$

$$\lim_{\varepsilon} \int_{\mathbf{R}} \varphi(x) \left(\int_{\varepsilon<|xi|<1/\varepsilon} e^{-2\pi i x\xi} \frac{d\xi}{\xi} \right) dx =$$

$$= -i \lim_{\varepsilon} \int_{\mathbf{R}} \varphi(x) \left(\int_{\varepsilon<|xi|<1/\varepsilon} \sin 2\pi x\xi \frac{d\xi}{\xi} \right) dx =$$

Last inner integral is known to be uniformly bounded in ε, x and has limit $\pi\mathrm{sign}(x)$, so the Fourier transform of $p.v.\frac{1}{x}$ is $-i\pi\mathrm{sign}(\xi)$.

The Fourier transform of a distribution with compact support

- If $f \in \mathcal{D}(\mathbf{R^d})$, then $\hat{f} \in \mathcal{S}(\mathbf{R^d})$. In fact something much more precise can be said.
- Note that $\hat{f}(\xi)$ makes sense for $z \in \mathbf{C^d}$,

$$\hat{f}(z) = \int_{\mathbf{R^d}} f(x) e^{-2\pi i z \cdot x}\, dx.$$

  and it is an entire function in $\mathbf{C^d}$ (in particular it cannot have compact support in $\mathbf{R^d}$).
- $u \in \mathcal{E}'(\mathbf{R^d})$, as it is capable to act on C^∞- functions not necessarily with compact support, may consider the entire function

$$h(z) = \langle u_x, e^{-2\pi i z \cdot x} \rangle,$$

  which is formally $\hat{u}(x)$ for $x \in \mathbf{R^d}$.

- One can check that the two definitions of $\hat{u}, u \in \mathcal{E}'(\mathbf{R^d})$ agree, that is,

$$\langle u, \hat{\varphi} \rangle = \int_{\mathbf{R^d}} h(x)\varphi(x).$$

- This means that for $u \in \mathcal{E}'(\mathbf{R^d})$, $\hat{u}$ is in fact the restriction to $\mathbf{R^d}$ of an entire function. Moreover, it is easy to see that $\hat{u} \in \mathcal{B}(\mathbf{R^d})$.
- Two Paley-Wiener theorems characterize exactly the class of entire functions that are Fourier transforms of $\mathcal{D}(\mathbf{R^d})$ and $\mathcal{E}'(\mathbf{R^d})$.

Convolution of a function with a distribution

- Want to define convolutions, $g * f(x) = \int g(x-y)f(y)dy$.
- Want to replace f by a general distribution u we should define

$$(g * u)(x) = \langle u_y, g(x-y) \rangle,$$

- This makes sense in three cases, the resulting function being

  $\mathcal{D}(\mathbf{R^d}) * \mathcal{D}'(\mathbf{R^d}) \subset \mathbf{C}^\infty(\mathbf{R^d}), \mathcal{S}(\mathbf{R^d}) * \mathcal{S}'(\mathbf{R^d}) \subset \mathbf{C}^\infty(\mathbf{R^d})$
  $C^\infty(\mathbf{R^d}) * \mathcal{E}'(\mathbf{R^d}) \subset \mathbf{C}^\infty(\mathbf{R^d}), \mathcal{D}(\mathbf{R^d}) * \mathcal{E}'(\mathbf{R^d}) \subset \mathcal{D}(\mathbf{R^d})$
- In fact in the second case, $\varphi * u \in \mathcal{B}(\mathbf{R^d})$.
- All rules that make sense hold:

  1. Convolution is continuous in both variables.
  2. $D^\alpha(g * u) = (D^\alpha g) * u = g * D^\alpha u$
  3. $\widehat{\varphi * u} = \hat{\varphi}\hat{u}, \varphi \in \mathcal{S}(\mathbf{R^d}), \mathbf{u} \in \mathcal{S}'(\mathbf{R^d})$
  4. $\widehat{\varphi u} = \hat{\varphi} * \hat{u}, \varphi \in \mathcal{S}(\mathbf{R^d}), \mathbf{u} \in \mathcal{S}'(\mathbf{R^d})$

Convolution of two distributions

- From

$$\langle g * f, \varphi \rangle = \int \int g(x-y) f(y) \varphi(x) dx\, dy =$$

$$= \int (\int g(z) \varphi(y+z) dz) f(y) dy = \int (\int f(y) \varphi(y+z) dy) g(z) dz$$

- Should define $\langle u * v, \varphi \rangle = \langle u_z, \langle v_y, \varphi(y+z) \rangle \rangle$, or $\langle u * v, \varphi \rangle = \langle v_y, \langle u_z, \varphi(y+z) \rangle \rangle$.
- To make sense, one of the distributions must have compact support. Fortunately the two definitions agree and defines $u * v \in \mathcal{D}'(\mathbf{R^d})$.
- $\delta * u = u$ for all distributions u,
- $D^\alpha (u * v) = (D^\alpha u) * v = u * D^\alpha v$.
- $\mathcal{E}'(\mathbf{R^d}) * \mathcal{S}'(\mathbf{R^d}) \subset \mathcal{S}'(\mathbf{R^d})$ and $\widehat{u * v} = \hat{u} \hat{v}$.

Translation invariant operators in test spaces

- With the language of distributions and tempered distributions one can state a number of representation theorems for continuous operators in spaces of distributions invariant by translations.
- Assume that $T : \mathcal{D}(\mathbf{R^d}) \to \mathbf{C}(\mathbf{R^d})$ is t.i.o. Then $\varphi \to T(\varphi)(0)$ is a distribution, and using translation invariance we find that $T(\varphi) = \varphi * u$ for $u \in \mathcal{D}'(\mathbf{R^d})$.
- It will take values in $\mathcal{D}(\mathbf{R^d})$ iff $u \in \mathcal{E}'(\mathbf{R^d})$, Thus, convolution by $u \in \mathcal{E}'(\mathbf{R^d})$ is the general form of a continuous t.i.o. from $\mathcal{D}(\mathbf{R^d})$ to itself.
- Convolution by $u \in \mathcal{E}'(\mathbf{R^d})$ is the general form of a t.i.o. from $C^\infty(\mathbf{R^d})$ to itself.
- Convolution by $u \in \mathcal{S}'(\mathbf{R^d})$ is the general form of a t.i.o. from $\mathcal{S}(\mathbf{R^d})$ to $C^\infty(\mathbf{R^d})$. It will take $\mathcal{S}(\mathbf{R^d})$ to itself iff $\hat{u} \in \mathcal{B}(\mathbf{R^d})$.

Translation invariant operators in spaces of distributions

- A basic fact of the spaces $C^\infty(\mathbf{R^d}), \mathcal{D}(\mathbf{R^d})$ and $\mathcal{S}(\mathbf{R^d})$ is that they are *reflexive*. This means that they equal their second dual.
- In other words, if $\omega : \mathcal{D}'(\mathbf{R^d}) :\to \mathbf{C}$ is a continuous linear functional, then there exists $\varphi \in \mathcal{D}(\mathbf{R^d})$ such that $\omega(u) = \langle u, \varphi \rangle$.
- Analogously, every continuous linear functional on $\mathcal{E}'(\mathbf{R^d})$ is given by testing on some $\varphi \in \mathcal{S}(\mathbf{R^d})$ and every continuous linear functional on $\mathcal{S}'(\mathbf{R^d})$ is given by testing on some $\varphi \in \mathcal{S}(\mathbf{R^d})$.
- Using this it is easy to prove that the general form of a continuous t.i.o. $T : \mathcal{D}'(\mathbf{R^d}) \to \mathcal{D}'(\mathbf{R^d})$ or $T : \mathcal{E}'(\mathbf{R^d}) \to \mathcal{E}'(\mathbf{R^d})$ is convolution by some $u\mathcal{E}'(\mathbf{R^d})$.
- Convolution by $u \in \mathcal{S}'(\mathbf{R^d})$ with $\hat{u} \in \mathcal{B}(\mathbf{R^d})$ is the general form of a continuous t.i.o. from $\mathcal{S}'(\mathbf{R^d})$ to $\mathcal{S}'(\mathbf{R^d})$.

Hormander's theorem

- If X, Y are some spaces of tempered distributions in which $\mathcal{S}(\mathbf{R^d})$ is dense, like all $L^p(\mathbf{R^d})$ spaces, every continuous t.i.p T from X to Y is given by convolution with some $u \in \mathcal{S}'(\mathbf{R^d})$.
- Indeed it will commute with convolution with L^1 functions so $T(\varphi * \psi) = \varphi * T(\psi) = T(\varphi) * \psi$, hence

$$\hat{\varphi} * \widehat{T\psi} = \hat{\psi}\widehat{T\varphi},$$

  hence $\widehat{T\psi} = m\hat{\psi}, m = \frac{\widehat{T\varphi}}{\hat{\varphi}}$.
- Choosing $\varphi(x) = e^{-|x|}$, $\hat{\varphi}(\xi)$ is the Poisson kernel whose inverse has slow growth. Then $m \in \mathcal{S}'(\mathbf{R^d})$

Fundamental solutions

- If $T : \mathcal{D}'(\mathbf{R^d}) :\to \mathcal{D}'(\mathbf{R^d})$ is t.i.o. we say that $E \in \mathcal{D}'(\mathbf{R^d})$ is a *fundamental solution* if $T(E) = \delta_0$.
- In this case $T(E * f) = T(E) * f = \delta * f = f$ whenever $E * f$ makes sense.
- Malgrange-Ehrenpreis theorem: every linear constant coefficient operator $P(D)$ has a fundamental solution.
- Note that if T is a t.i.o. operator in $\mathcal{S}'(\mathbf{R^d})$, then is convolution with $u \in \mathcal{S}'(\mathbf{R^d})$ with $\hat{u} = m \in \mathcal{B}(\mathbf{R^d})$, so $1 = m\hat{E}$. If $\frac{1}{m} \in \mathcal{S}'(\mathbf{R^d})$, then the tempered distribution with $\hat{E} = \frac{1}{m}$ is the fundamental solution.
- For the laplacian Δ, $m(\xi) = -4\pi^2|\xi|^2$ and $E(x) = c_d|x|^{2-d}$ when $d > 2$ and $E(x) = c_2 \log|x|$ when $d = 2$.
- It then follows that $\Delta(E * f) = f$ in the sense of distributions for every $f \in \mathcal{S}(\mathbf{R^d})$.
- Weyl's lemma: if $f \in C^\infty(\mathbf{R^d})$ and $\Delta u = f$ in the sense of distributions, then $u \in C^\infty(\mathbf{R^d})$ and $\Delta u = f$ in the classical sense.

The Poisson summation formula. An unified language for the Fourier transform

- $\frac{1}{a}\widehat{\Delta_{\frac{1}{a}}} = \Delta_a$ as tempered distributions. This means exactly that

$$\sum_n \varphi(na) = \frac{1}{a}\sum_n \widehat{\varphi}(\frac{n}{a}), \varphi \in \mathcal{S}(\mathbf{R^d}).$$

- Replacing φ by $\tau_{-x}\varphi$ we get

$$\sum_n \varphi(x - na) = \frac{1}{a}\sum_n \widehat{\varphi}(\frac{n}{a})e^{2\pi i x \frac{n}{a}},$$

or interchanging φ and $\hat{\varphi}$,

$$\sum_n \hat{\varphi}(x - na) = \frac{1}{a}\sum_n \varphi(na)e^{-2\pi i x \frac{n}{a}}.$$

- This is known as *Poisson's summation formula* and if fact holds for a much larger class of functions.

A unified language

- Assume that f is a a-periodic function integrable in one period. It has a formal series $\sum_n c_n(f)e^{2\pi i \frac{n}{a} x}$, not converge in general to f.
- Let us look at f as a tempered distribution and let us compute its Fourier transform.

$$\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle = \int_{\mathbf{R}} f(x)\hat{\varphi}(x)\, dx = \int_0^a f(x) \sum_n \hat{\varphi}(x - na) dx =$$

$$= \frac{1}{a} \int_0^a f(x) \sum_n \varphi(\frac{n}{a}) e^{-2\pi i x \frac{n}{a}} = \sum_n c_n(f) \varphi(\frac{n}{a})$$

- This means that as a tempered distribution

$$\hat{f} = \sum_n c_n \delta_{\frac{n}{a}}$$

the sum being convergent in $\mathcal{S}'(\mathbf{R^d})$.
- By applying the inverse Fourier transform we find that

$$f = \sum_n c_n(f) e^{2\pi i \frac{n}{a} x} \text{ in } \mathcal{S}'(\mathbf{R^d}).$$