Algorithmic market making for options

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Introduction
What is a market maker?

- A market maker is a liquidity provider. He/she provides bid and ask prices for a list of assets to other market participants.
- Today, market makers are often replaced by market making algorithms.
What is a market maker?

- A market maker is a liquidity provider. He / she provides bid and ask prices for a list of assets to other market participants.
- Today, market makers are often replaced by market making algorithms.

A market maker faces a complex optimization problem

- Makes money out of buying low and selling high (bid-ask spread).
- Faces the risk that the price moves adversely without him/her being able to unwind his position rapidly enough.
<table>
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<th>Literature: a bit of history</th>
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<tr>
<td>• Ho and Stoll (1981)</td>
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<td>• Grossman and Miller (1988)</td>
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Literature: a bit of history

- Ho and Stoll (1981)
- Grossman and Miller (1988)

New interest 20 years later

Avellaneda-Stoikov: a first model
Avellaneda-Stoikov modelling framework

- One asset with reference price process (mid-price) \((S_t)_t:\n
\[ dS_t = \sigma \, dW_t. \]
Avellaneda-Stoikov modelling framework

- One asset with reference price process (mid-price) \((S_t)_t\):

\[ dS_t = \sigma dW_t. \]

- Bid and ask prices of the MM denoted respectively

\[ S_t^b = S_t - \delta_t^b \] and \[ S_t^a = S_t + \delta_t^a. \]
Avellaneda-Stoikov modelling framework

- One asset with reference price process (mid-price) \((S_t)_t\):
  \[
dS_t = \sigma dW_t.
  \]

- Bid and ask prices of the MM denoted respectively
  \[
  S^b_t = S_t - \delta^b_t \quad \text{and} \quad S^a_t = S_t + \delta^a_t.
  \]

- Point processes \(N^b\) and \(N^a\) (indep. of \(W\)) for the transactions (size \(z = 1\)). Inventory \((q_t)_t\):
  \[
dq_t = zdN^b_t - zdN^a_t.
  \]
The intensities of $N^b$ and $N^a$ depend on the distance to the reference price:

$$\lambda^b_t = \Lambda^b(\delta^b_t) \text{ and } \lambda^a_t = \Lambda^a(\delta^a_t).$$

$\Lambda^b$, $\Lambda^a$ decreasing. Avellaneda and Stoikov suggested

$$\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}.$$
• The intensities of $N^b$ and $N^a$ depend on the distance to the reference price:

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$$\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}.$$  

• Cash process $(X_t)_t$:

$$dX_t = zS_t^a dN^a_t - zS_t^b dN^b_t = -S_t dq_t + \delta^a_t zdN^a_t + \delta^b_t zdN^b_t.$$
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$$\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}.$$

Cash process $(X_t)_t$:

$$dX_t = zS^a_t dN^a_t - zS^b_t dN^b_t = -S_t dq_t + \delta^a_t zdN^a_t + \delta^b_t zdN^b_t.$$

Three state variables: $X$ (cash), $q$ (inventory), and $S$ (price).
Avellaneda-Stoikov objective function and HJB equation

**CARA objective function**

\[
\sup_{(\delta^a_t, \delta^b_t) \in A} \mathbb{E} \left[ -\exp \left( -\gamma (X_T + q_T S_T) \right) \right],
\]

where \( \gamma \) is the absolute risk aversion parameter, \( T \) a time horizon, and \( A \) the set of predictable processes bounded from below.
Avellaneda-Stoikov objective function and HJB equation

**CARA objective function**

\[
\sup_{(\delta^a_t, (\delta^b_t)_{t \in \mathcal{A}}) \in \mathcal{A}} \mathbb{E} \left[ - \exp \left( -\gamma(X_T + q_T S_T) \right) \right],
\]

where \( \gamma \) is the absolute risk aversion parameter, \( T \) a time horizon, and \( \mathcal{A} \) the set of predictable processes bounded from below.

**An (a priori) awful Hamilton-Jacobi-Bellman**

\[
(HJB) \quad 0 = \partial_t u(t, x, q, S) + \frac{1}{2} \sigma^2 \partial^2_{SS} u(t, x, q, S) \\
+ \sup_{\delta^b} \Lambda^b(\delta^b) \left[ u(t, x - zS + z\delta^b, q + z, S) - u(t, x, q, S) \right] \\
+ \sup_{\delta^a} \Lambda^a(\delta^a) \left[ u(t, x + zS + z\delta^a, q - z, S) - u(t, x, q, S) \right]
\]

with final condition:

\[
u(T, x, q, S) = -\exp \left( -\gamma(x + qS) \right).
\]
A rigorous analysis

Solution of the Avellaneda-Stoikov model

A rigorous analysis

Solution of the Avellaneda-Stoikov model


When risk limits are set, solving the AS model with exponential intensities boils down to solving a system of linear ordinary differential equations!
Market making: an interesting research strand
Many extensions of the initial one-asset model

- Multi-asset framework.
- General intensities (e.g. logistic).
- Variable RFQ sizes.
- Different objective functions (mean-variance-like criterion).
- Client tiering.
- Adverse selection.
- Drift / signal / alpha.
- Access to liquidity pools (exchange / IDB - for some asset classes).
- Market and limit orders (not relevant for all asset classes).
- ...


Papers by Cartea, Jaimungal et al.

Figure 1: A nice book dealing with market making
Papers by Guilbaud and Pham

Multi-asset market making

Extensions to multi-asset portfolios


Figure 2: Another nice book
The problem

The number of equations to solve typically grows exponentially with the number of assets.
### The problem

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### Attempts to solve it

- Other works in progress.

Our paper on options is inspired by the first approach.
Multi-asset market making

The problem
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Attempts to solve it
## Multi-asset market making

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## Multi-asset market making

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Asset classes

Extensions to derivatives


- Baldacci, Bergault and Guéant. Algorithmic market making for options. 2020. (On ArXiv, in revision)
Asset classes

Extensions to derivatives

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Relevant models should handle several option contracts and tackle the question of Δ-hedging / trading in the underlying asset.
Option market making: the model
The market

**Asset price dynamics under \( \mathbb{P} \)**

\[
\left\{
\begin{array}{l}
    dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S \\
    d\nu_t = a_P(t, \nu_t) dt + \xi \sqrt{\nu_t} dW_t^\nu.
\end{array}
\right.
\]
The market

Asset price dynamics under $\mathbb{P}$

\[
\begin{align*}
    dS_t &= \mu S_t \, dt + \sqrt{\nu_t} S_t \, dW_t^S \\
    d\nu_t &= a_\mathbb{P}(t, \nu_t) \, dt + \xi \sqrt{\nu_t} \, dW_t^\nu.
\end{align*}
\]

Asset price dynamics under $\mathbb{Q}$ (pricing measure) - $r = 0$

\[
\begin{align*}
    dS_t &= \sqrt{\nu_t} S_t \, d\hat{W}_t^S \\
    d\nu_t &= a_\mathbb{Q}(t, \nu_t) \, dt + \xi \sqrt{\nu_t} \, d\hat{W}_t^\nu.
\end{align*}
\]
Another one-factor model can be chosen (e.g. Bergomi). Two-factor models are also possible: they increase the dimensionality of the problem by 1.
The market

The options

- We consider $N \geq 1$ European options written on the above asset.
- For $i = 1, \ldots, N$:
  - Maturity of the $i$-th option: $T^i$
  - Price process of the $i$-th option: $(O_t^i)_{t \in [0, T^i]}$
The market

The options

- We consider $N \geq 1$ European options written on the above asset.
- For $i = 1, \ldots, N$:
  - Maturity of the $i$-th option: $T^i$
  - Price process of the $i$-th option: $(O^i_t)_{t \in [0, T^i]}$

Partial differential equation

$O^i_t = O^i(t, S_t, \nu_t)$ where

$$0 = \partial_t O^i(t, S, \nu) + a_Q(t, \nu)\partial_\nu O^i(t, S, \nu) + \frac{1}{2} \nu S^2 \partial^2_{SS} O^i(t, S, \nu)$$

$$+ \rho \xi \nu S \partial^2_{\nu S} O^i(t, S, \nu) + \frac{1}{2} \xi^2 \nu \partial^2_{\nu \nu} O^i(t, S, \nu).$$
Requests

Distribution of requests

- Requests on option $i$ arrive with intensities $\lambda^{i,b}_{\text{request}}$ and $\lambda^{i,a}_{\text{request}}$.
- Request sizes for option $i$ are distributed according to probability measures $\mu^{i,b}(dz)$ and $\mu^{i,a}(dz)$.
- Bid and ask prices answered for option $i$ (transaction of size $z$) if the transaction does not violate risk limits:
  
  $$\mathcal{O}^i_t - \delta^{i,b}_t(z) \quad \text{and} \quad \mathcal{O}^i_t + \delta^{i,a}_t(z).$$

- Probabilities of trading:
  
  $$f^{i,b}(\delta^{i,b}_t(z)) \quad \text{and} \quad f^{i,a}(\delta^{i,a}_t(z)).$$
Inventory in the options

Resulting dynamics of the inventory process

\[ dq^i_t = \int_{\mathbb{R}^*_+} zN^i,b (dt, dz) - \int_{\mathbb{R}^*_+} zN^i,a (dt, dz), \]

where \( N^i,b \) and \( N^i,a \) are marked point processes with kernels:

\[ \nu^{i,b}_t (dz) = \lambda^{i,b}_{\text{request}} f^{i,b}_t (\sigma^{i,b}_t (z)) \mathbb{1}_{\{q_t - ze^i \in Q\}} \mu^{i,b}_t (dz), \]

\[ \Lambda^{i,b}_t (\sigma^{i,b}_t (z)) \]

\[ \nu^{i,a}_t (dz) = \lambda^{i,a}_{\text{request}} f^{i,a}_t (\sigma^{i,a}_t (z)) \mathbb{1}_{\{q_t - ze^i \in Q\}} \mu^{i,a}_t (dz), \]

\[ \Lambda^{i,a}_t (\sigma^{i,a}_t (z)) \]
Inventory in the underlying asset

**Δ-hedging**

The market maker ensures perfect Δ-hedging where

$$\Delta_t = \sum_{i=1}^{N} \partial S^O_i(t, S_t, \nu_t) q_t^i.$$
Inventory in the underlying asset

\[ \Delta_t = \sum_{i=1}^{N} \partial_S O^i(t, S_t, \nu_t) q^i_t. \]

Continuous trading is our real assumption:

- The assumption of perfect \( \Delta \)-hedging can in fact be relaxed.
- One can hedge part of the vega by trading the underlying asset. → See the appendix of our paper on ArXiv.
Cash dynamics and Mark-to-Market value of the portfolio

**Cash dynamics**

\[
\begin{align*}
    dX_t &= \sum_{i=1}^{N} \left( \int_{\mathbb{R}_+^*} z \left( \delta^{i,b}_t (z) N^{i,b}_t (dt, dz) + \delta^{i,a}_t (z) N^{i,a}_t (dt, dz) \right) - O^i_t dq^i_t \right) \\
    &\quad + S_t d\Delta_t + d\langle \Delta, S \rangle_t.
\end{align*}
\]
Cash dynamics

\[ dX_t = \sum_{i=1}^{N} \left( \int_{\mathbb{R}^+} z \left( \delta^{i,b}_t(z) N^{i,b} (dt, dz) + \delta^{i,a}_t(z) N^{i,a} (dt, dz) \right) - \mathcal{O}^i_t dq^i_t \right) + S_t d\Delta_t + d\langle \Delta, S \rangle_t. \]

Mark-to-Market value of the portfolio

\[ V_t = X_t - \Delta_t S_t + \sum_{i=1}^{N} q^i_t \mathcal{O}^i_t. \]
Dynamics of the Mark-to-Market value of the portfolio

Dynamics of the MtM value

\[ dV_t = \sum_{i=1}^{N} \int_{\mathbb{R}^+} z \left( \delta_t^{i,b}(z) N^{i,b}(dt, dz) + \delta_t^{i,a}(z) N^{i,a}(dt, dz) \right) + \sum_{i=1}^{N} q_t^{i} dO_t^{i} - \Delta_t dS_t \]
Dynamics of the MtM value

\[ dV_t = \sum_{i=1}^{N} \int_{\mathbb{R}^*_+} z \left( \delta_{t}^{i,b}(z) N^{i,b}(dt, dz) + \delta_{t}^{i,a}(z) N^{i,a}(dt, dz) \right) + \sum_{i=1}^{N} q^i_t dO^i_t - \Delta_t dS_t \]

\[ = \sum_{i=1}^{N} \int_{\mathbb{R}^*_+} z \left( \delta_{t}^{i,a}(z) N^{i,a}(dt, dz) + \delta_{t}^{i,b}(z) N^{i,b}(dt, dz) \right) \]

\[ + \sum_{i=1}^{N} q^i_t \partial_\nu O^i(t, S_t, \nu_t) (a_P(t, \nu_t) - a_Q(t, \nu_t)) dt \]

\[ + \sqrt{\nu_t \xi} q^i_t \partial_\nu O^i(t, S_t, \nu_t) dW^\nu_t. \]
Introducing vegas

Vega of the $i$-th option

$$\nu^i_t := \partial_{\sqrt{\nu}} O^i(t, S_t, \nu_t) = 2\sqrt{\nu_t} \partial_{\nu} O^i(t, S_t, \nu_t).$$
Introducing vegas

Vega of the $i$-th option

$$\mathcal{V}_t^i := \partial_{\sqrt{\nu}} O^i(t, S_t, \nu_t) = 2\sqrt{\nu_t} \partial_{\nu} O^i(t, S_t, \nu_t).$$

Simplified dynamics of the MtM value

$$d\mathcal{V}_t = \sum_{i=1}^{N} \int_{\mathbb{R}_+^*} z \left( \delta^i, a(z) N^i, a(dt, dz) + \delta^i, b(z) N^i, b(dt, dz) \right)$$

$$+ \sum_{i=1}^{N} q^i \mathcal{V}_t \frac{a_P(t, \nu_t) - a_Q(t, \nu_t)}{2\sqrt{\nu_t}} dt + \sum_{i=1}^{N} \frac{\xi}{2} q^i \mathcal{V}_t dW^\nu_t.$$
Option market making:
optimization problem,
assumptions, and approximations
Objective function: risk-adjusted expectation

\[
\sup_{\delta \in \mathcal{A}} \mathbb{E} \left[ V_T - \frac{\gamma}{2} \int_0^T \left( \sum_{i=1}^N \frac{\xi}{2} q_t^i v_t^i \right)^2 dt \right].
\]

for \( \gamma \) a risk aversion parameter and \( T \) a time horizon such that \( T < \min_{i} T^i \).
Objective function

Objective function: risk-adjusted expectation

\[
\sup_{\delta \in \mathcal{A}} \mathbb{E} \left[ V_T - \frac{\gamma}{2} \int_0^T \left( \sum_{i=1}^N \frac{\xi}{2} q_i \nu_t^i \right)^2 dt \right].
\]

for \( \gamma \) a risk aversion parameter and \( T \) a time horizon such that \( T < \min_i T^i \).

\[
\sup_{\delta \in \mathcal{A}} \mathbb{E} \left[ \int_0^T \sum_{i=1}^N \left( \sum_{j=a,b} \int_{\mathbb{R}_+} z \delta_t^{i,j}(z) \Lambda^{i,j}(\delta_t^{i,j}(z)) \mathbb{1}_{\{q_t \pm jz \in \mathcal{Q}\}} \mu^{i,j}(dz) \right) dt - \frac{\gamma \xi^2}{8} \int_0^T \left( \sum_{i=1}^N q_t \nu_t^i \right)^2 dt \right],
\]

where \( \pm_b = + \) and \( \pm_a = - \).
Value function

\[
  u : (t, S, \nu, q) \in [0, T] \times \mathbb{R}^2_+ \times Q \mapsto u(t, S, \nu, q)
\]
given by

\[
  u(t, S, \nu, q) = \sup_{(\delta_s)_{s \in [t, T]} \in \mathcal{A}_t} \mathbb{E}_{(t, S, \nu, q)} \left[ \int_t^T \sum_{i=1}^N \left( \left( \sum_{j=a,b} \int_{\mathbb{R}^*_+} z \delta^i j(z) \Lambda^i j \left( \delta^i j(z) \right) \mathbb{1}_{\{q_s \pm j ze \in \mathcal{Q}\}} \mu^i j \left( dz \right) \right) \right. \\
  \left. + q_s \nu^s \left( \frac{a_P(s, \nu_s) - a_Q(s, \nu_s)}{2 \sqrt{\nu_s}} \right) ds - \frac{\gamma \xi^2}{8} \int_t^T \left( \sum_{i=1}^N q_s \nu^i_s \right)^2 ds \right]
\]
Value function

\[ u : (t, S, \nu, q) \in [0, T] \times \mathbb{R}^+ \times Q \mapsto u(t, S, \nu, q) \]
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u(t, S, \nu, q) = \sup_{(\delta_s)_{s \in [t, T]} \in \mathcal{A}_t} \mathbb{E}_{(t, S, \nu, q)} \left[ \int_t^T \sum_{i=1}^N \left( \sum_{j=a, b} \int_{\mathbb{R}^+} z \delta_s^{i,j}(z) \Lambda_{i,j}(\delta_s^{i,j}(z)) \mathbbm{1}_{\{q_s - \pm jzei \in \mathcal{Q}\}} \mu^{i,j}(dz) \right) \right. \\
+ q_s \mathcal{V}_s \left( \frac{a_p(s, \nu_s) - a_Q(s, \nu_s)}{2\sqrt{\nu_s}} \right) ds - \frac{\gamma \xi^2}{8} \int_t^T \left( \sum_{i=1}^N q_i \mathcal{V}_s^i \right)^2 ds \]

The problem is written in (space) dimension \( N + 2 \): it is a priori untractable!
Assumption 1

We approximate the vega of each option over \([0, T]\) by its value at time \(t = 0\), namely 
\[
\mathcal{V}_t^i = \mathcal{V}_0^i =: \mathcal{V}^i, \quad i = 1, \ldots, N.
\]
Assumption 1
We approximate the vega of each option over \([0, T]\) by its value at time \(t = 0\), namely \(\mathcal{V}_t^i = \mathcal{V}_0^i =: \mathcal{V}^i, \quad i = 1, \ldots, N\).

Assumption 2
Authorized inventories correspond to vega risk limits:

\[
Q = \left\{ q \in \mathbb{R}^N \left| \sum_{i=1}^{N} q^i \mathcal{V}^i \in [-\mathcal{V}, \mathcal{V}] \right. \right\}, \quad \text{with } \mathcal{V} \in \mathbb{R}^{+*}.
\]
Portfolio vega

\[ \nu_t^\pi := \sum_{i=1}^{N} q_i^t \nu_i. \]
Change of variables

Portfolio vega

\[ V^\pi_t := \sum_{i=1}^{N} q^i_t V^i. \]

Optimal control problem

\[
\nu(t, \nu, V^\pi) = \sup_{(\delta_s)_{s \in [t, T]} \in \mathcal{A}_t} \mathbb{E}(t, \nu, V^\pi)
\]

\[
\left[ \int_t^T \left( \left( \sum_{i=1}^{N} \sum_{j=a,b} \int_{\mathbb{R}_+^*} z \delta^{i,j}(z) \Lambda^{i,j}(\delta^{i,j}_s(z)) \mathbb{1}_{|V^\pi_s \pm \nu^i_s z V^i| \leq V^{\mu^{i,j}} (dz)} \right) 
+ V^\pi_s \left( \frac{a_P(s, \nu_s) - a_Q(s, \nu_s)}{2\sqrt{\nu_s}} - \frac{\gamma \xi^2}{8 V^2_s} \right) \right] ds \right].
\]
Low-dimensional HJB equation

The associated Hamilton-Jacobi-Bellman equation is:

\[
0 = \partial_t v(t, \nu, V^\pi) + a_p(t, \nu) \partial_\nu v(t, \nu, V^\pi) + \frac{1}{2} \nu \xi^2 \partial_{\nu, \nu} v(t, \nu, V^\pi) \\
+ \nu^\pi \frac{a_p(t, \nu) - a_q(t, \nu)}{2\sqrt{\nu}} - \frac{\gamma \xi^2}{8} V^\pi 2 \\
+ \sum_{i=1}^{N} \sum_{j=a,b} \int_{\mathbb{R}^*_+} z 1_{|V^\pi \pm j z V^i| \leq V} H^{i,j}(v(t, \nu, V^\pi) - v(t, \nu, V^\pi \pm j z V^i)) \mu^{i,j}(dz),
\]

with final condition \( v(T, \nu, V^\pi) = 0 \), where

\[
H^{i,j}(p) := \sup_{\delta^{i,j} \geq \delta_\infty} \Lambda^{i,j}(\delta^{i,j})(\delta^{i,j} - p).
\]
Low-dimensional HJB equation

The associated Hamilton-Jacobi-Bellman equation is:

\[ 0 = \partial_t v(t, \nu, \mathcal{V}_\pi) + a_P(t, \nu) \partial_\nu v(t, \nu, \mathcal{V}_\pi) + \frac{1}{2} \nu \xi^2 \partial_{\nu \nu} v(t, \nu, \mathcal{V}_\pi) \]

\[ + \mathcal{V}_\pi \frac{a_P(t, \nu) - a_Q(t, \nu)}{2 \sqrt{\nu}} - \frac{\gamma \xi^2}{8} \mathcal{V}_\pi^2 \]

\[ + N \sum_{i=1}^{N} \sum_{j=a,b} \int_{\mathbb{R}_+^*} z 1_{|\nu \mathcal{V}_\pi \pm j z \mathcal{V}_i| \leq \mathcal{V}_\pi} H^{i,j}(v(t, \nu, \mathcal{V}_\pi) - v(t, \nu, \mathcal{V}_\pi \pm j z \mathcal{V}_i) \mathcal{V}_i) \mu^{i,j}(dz), \]

with final condition \( v(T, \nu, \mathcal{V}_\pi) = 0 \), where

\[ H^{i,j}(p) := \sup_{\delta^{i,j} \geq \delta_\infty} \Lambda^{i,j}(\delta^{i,j})(\delta^{i,j} - p). \]

This equation in (space) dimension 2 can be solved numerically on a grid with a Euler scheme and linear interpolation.
Once the value function is known, the optimal mid-to-bid and ask-to-mid associated with the $N$ options, are given by the following formula:

$$\delta_{t,j}^*(z) = \max \left( \delta_{\infty}, \left( \Lambda_{i,j} \right)^{-1} \left( -H_{i,j}' \left( \frac{v(t, \nu_t, V_{t_+}^\pi) - v(t, \nu_t, V_{t_-}^\pi \pm j z V^i)}{z} \right) \right) \right).$$
Change of variables when $a_{P} = a_{Q}$

If $a_{P} = a_{Q}$, then $\nu(t, \nu, \nu^\pi) = w(t, \nu^\pi)$ where $w$ is solution of the simpler Hamilton-Jacobi-Bellman:

$$0 = \partial_t w(t, \nu^\pi) - \frac{\gamma \xi^2}{8} \nu^\pi^2$$

$$+ \sum_{i=1}^{N} \sum_{j=a, b} \int_{\mathbb{R}^*_+} z 1_{|\nu^\pi \pm j z \nu^i| \leq \nu} H^{i,j} \left( \frac{w(t, \nu^\pi) - w(t, \nu^\pi \pm j z \nu^i)}{z} \right) \mu^{i,j}(dz),$$

with final condition $w(T, \nu^\pi) = 0$. 

The problem is now in (space) dimension 1.
Change of variables when \( a_P = a_Q \)

If \( a_P = a_Q \), then \( \nu(t, \nu, \nu^\pi) = w(t, \nu^\pi) \) where \( w \) is solution of the simpler Hamilton-Jacobi-Bellman:

\[
0 = \partial_t w(t, \nu^\pi) - \frac{\gamma \xi^2}{8} \nu^\pi^2 + \sum_{i=1}^{N} \sum_{j=a,b} \int_{\mathbb{R}_+^*} z 1_{|\nu^\pi \pm jz\nu^i| \leq \nu} H^{i,j}(w(t, \nu^\pi) - w(t, \nu^\pi \pm jz\nu^i)) \mu^{i,j}(dz),
\]

with final condition \( w(T, \nu^\pi) = 0. \)

The problem is now in (space) dimension 1.
Numerical results
Model parameters

- Stock price at time $t = 0$: $S_0 = 10 \, \text{€}$.
- Instantaneous variance at time $t = 0$: $\nu_0 = 0.0225 \, \text{year}^{-1}$.
- Heston model with $a_P(t, \nu) = \kappa_P(\theta_P - \nu)$ where $\kappa_P = 2 \, \text{year}^{-1}$ and $\theta_P = 0.04 \, \text{year}^{-1}$, and $a_Q(t, \nu) = \kappa_Q(\theta_Q - \nu)$ where $\kappa_Q = 3 \, \text{year}^{-1}$ and $\theta_Q = 0.0225 \, \text{year}^{-1}$.
- Volatility of volatility: $\xi = 0.2 \, \text{year}^{-1}$.
- Spot-variance correlation: $\rho = -0.5$. 
Strikes and maturities

\[ \mathcal{K} = \{8 \text{ €}, 9 \text{ €}, 10 \text{ €}, 11 \text{ €}, 12 \text{ €}\} \]

\[ \mathcal{T} = \{1 \text{ year, 1.5 years, 2 years, 3 years}\} \]

Intensities

\[ \Lambda_{i,j}(\delta) = \lambda_{\text{request}} + e^{\alpha + \beta V_i \delta}, \quad i \in \{1, \ldots, N\}, \quad j = a, b. \]

where \( \lambda_{\text{request}} = 17640 = 252 \times 30 \text{ year}^{-1} \), \( \alpha = 0.7 \), and \( \beta = 150 \text{ year}^{-1/2} \).

Size of transactions

\[ z_i = 5 \cdot 10^5 \text{ contracts} : \mu_i, b \] and \( \mu_i, a \) are Dirac masses.
Strikes and maturities

\[ \mathcal{K} = \{8 \text{ €}, 9 \text{ €}, 10 \text{ €}, 11 \text{ €}, 12 \text{ €}\} \]
\[ \mathcal{T} = \{1 \text{ year, 1.5 years, 2 years, 3 years}\}. \]

Intensities

\[ \Lambda_{i,j}^{\mathcal{K}}(\delta) = \frac{\lambda_{\text{request}}}{1 + e^{\alpha + \frac{\beta}{\nu} \delta}}, \quad i \in \{1, \ldots, N\}, \quad j = a, b. \]

where \( \lambda_{\text{request}} = 17640 = 252 \times 30 \text{ year}^{-1} \), \( \alpha = 0.7 \), and \( \beta = 150 \text{ year}^{\frac{1}{2}} \).
Options

**Strikes and maturities**

\[ \mathcal{K} = \{8 \, \text{€}, 9 \, \text{€}, 10 \, \text{€}, 11 \, \text{€}, 12 \, \text{€}\} \]

\[ \mathcal{T} = \{1 \, \text{year}, 1.5 \, \text{years}, 2 \, \text{years}, 3 \, \text{years}\}. \]

**Intensities**

\[ \Lambda^{i,j}(\delta) = \frac{\lambda_{\text{request}}}{1 + e^{\alpha \delta + \frac{\beta}{\nu} \delta}}, \quad i \in \{1, \ldots, N\}, \quad j = a, b. \]

where \( \lambda_{\text{request}} = 17640 = 252 \times 30 \, \text{year}^{-1} \), \( \alpha = 0.7 \), and \( \beta = 150 \, \text{year}^{\frac{1}{2}} \).

**Size of transactions**

\[ z^i = \frac{5 \times 10^5}{\mathcal{O}_0} \] contracts: \( \mu^{i,b} \) and \( \mu^{i,a} \) are Dirac masses.
**Figure 3:** Implied volatility surface associated with the above parameters.
Figure 4: Value function as a function of the portfolio vega for $\nu = 0.0225 - \gamma = 10^{-3} \欧元^{-1}$, $t=0$, $T=0.2$ days.
Figure 5: Optimal mid-to-bid quotes as a function of time for $\nu = 0.0225 - \gamma = 10^{-3} \text{ €}^{-1}$.
**Figure 6:** Optimal mid-to-bid quotes divided by option price for $K = 8$ and $\nu = 0.0225 - \gamma = 10^{-3} \; \varepsilon^{-1}$. 
Figure 7: Optimal mid-to-bid quotes divided by option price for $K = 9$ and $\nu = 0.0225 - \gamma = 10^{-3} \, \text{€}^{-1}$. 
Optimal bid quotes

Figure 8: Optimal mid-to-bid quotes divided by option price for $K = 10$ and $\nu = 0.0225 - \gamma = 10^{-3} \, \text{€}^{-1}$.
Figure 9: Optimal mid-to-bid quotes divided by option price for $K = 11$ and $\nu = 0.0225 - \gamma = 10^{-3} \, \text{€}^{-1}$. 
Optimal bid quotes

\[(K_i, T_i) = (12.0, 1.0) \quad \text{-- price} = 0.62, \text{vega} = 5.71\]
\[(K_i, T_i) = (12.0, 1.5) \quad \text{-- price} = 1.32, \text{vega} = 6.63\]
\[(K_i, T_i) = (12.0, 2.0) \quad \text{-- price} = 2.07, \text{vega} = 6.86\]
\[(K_i, T_i) = (12.0, 3.0) \quad \text{-- price} = 3.66, \text{vega} = 6.68\]

Figure 10: Optimal mid-to-bid quotes divided by option price for \(K = 12\) and \(\nu = 0.0225 - \gamma = 10^{-3} \, \text{€}^{-1}\).
Figure 11: Optimal bid quotes in terms of implied volatility for \( K = 8 \) and \( \nu = 0.0225 - \gamma = 10^{-3} \text{ } \varepsilon^{-1} \).
Figure 12: Optimal bid quotes in terms of implied volatility for $K = 9$ and $\nu = 0.0225 - \gamma = 10^{-3} \, \text{€}^{-1}$.
Figure 13: Optimal bid quotes in terms of implied volatility for $K = 10$ and $\nu = 0.0225 - \gamma = 10^{-3} \, \text{€}^{-1}$. 
Figure 14: Optimal bid quotes in terms of implied volatility for $K = 11$ and $\nu = 0.0225 - \gamma = 10^{-3} \text{ } €^{-1}$.
Figure 15: Optimal bid quotes in terms of implied volatility for $K = 12$ and $\nu = 0.0225 - \gamma = 10^{-3} \, \text{€}^{-1}$. 
Conclusive remarks

- Option market making is tractable using one- or two-factor stochastic volatility models.
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- Option market making is tractable using one- or two-factor stochastic volatility models.
- It is possible to go beyond the constant-vega approximation using a Taylor expansion around that approximation → see the appendix of our paper.
- A model with several underlying assets can easily be written. The feasibility of numerical approximation with grids depend on the global number of factors.
Thanks for your attention.
Do not hesitate to make remarks and ask questions.