Algorithmic market making for options

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Introduction

Market Making

What is a market maker?

- A market maker is a liquidity provider. He / she provides bid and ask prices for a list of assets to other market participants.
- Today, market makers are often replaced by market making algorithms.

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A market maker faces a complex optimization problem

- Makes money out of buying low and selling high (bid-ask spread).
- Faces the risk that the price moves adversely without him/her being able to unwind his position rapidly enough.

Literature

Literature: a bit of history

- Ho and Stoll (1981)
- Grossman and Miller (1988)

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- Ho and Stoll (1981)
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New interest 20 years later

- Avellaneda and Stoikov. High-frequency trading in a limit order book. Quantitative Finance, 2008.

Avellaneda-Stoikov: a first model

Avellaneda-Stoikov modelling framework

- One asset with reference price process (mid-price) $(S_t)_t$:

$$dS_t = \sigma dW_t.$$

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- Point processes N^b and N^a (indep. of W) for the transactions (size $z = 1$). Inventory $(q_t)_t$:

$$dq_t = zdN_t^b - zdN_t^a.$$

Avellaneda-Stoikov modelling framework (continued)

- The intensities of N^b and N^a depend on the distance to the reference price:

$$\lambda_t^b = \Lambda^b(\delta_t^b) \text{ and } \lambda_t^a = \Lambda^a(\delta_t^a).$$

Λ^b, Λ^a decreasing. Avellaneda and Stoikov suggested

$$\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}.$$

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- Cash process $(X_t)_t$:

$$dX_t = zS_t^a dN_t^a - zS_t^b dN_t^b = -S_t dq_t + \delta_t^a z dN_t^a + \delta_t^b z dN_t^b.$$

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Three state variables: X (cash), q (inventory), and S (price).

Avellaneda-Stoikov objective function and HJB equation

CARA objective function

$$\sup_{(\delta_t^a)_t,(\delta_t^b)_t \in \mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma(X_T + q_T S_T)\right)\right],$$

where γ is the absolute risk aversion parameter, T a time horizon, and $\mathcal{A}$ the set of predictable processes bounded from below.

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An (a priori) awful Hamilton-Jacobi-Bellman

$$\text{(HJB)} \qquad 0 = \partial_t u(t,x,q,S) + \frac{1}{2}\sigma^2\partial^2_{SS}u(t,x,q,S)$$

$$+\sup_{\delta^b}\Lambda^b(\delta^b)\left[u(t,x-zS+z\delta^b,q+z,S)-u(t,x,q,S)\right]$$

$$+\sup_{\delta^a}\Lambda^a(\delta^a)\left[u(t,x+zS+z\delta^a,q-z,S)-u(t,x,q,S)\right]$$

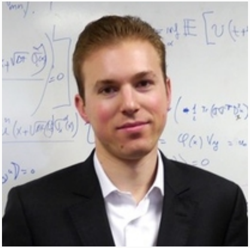
with final condition:

$$u(T,x,q,S) = -\exp\left(-\gamma(x+qS)\right).$$

A rigorous analysis

Solution of the Avellaneda-Stoikov model

- Guéant, Lehalle, and Fernandez-Tapia. Dealing with the Inventory Risk: A solution to the market making problem. MAFE, 2013.



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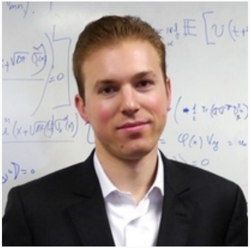

When risk limits are set, solving the AS model with exponential intensities boils down to solving a system of linear ordinary differential equations!

Market making: an interesting research strand

Literature

Many extensions of the initial one-asset model

- Multi-asset framework.
- General intensities (*e.g. logistic*).
- Variable RFQ sizes.
- Different objective functions (*mean-variance-like criterion*).
- Client tiering.
- Adverse selection.
- Drift / signal / alpha.
- Access to liquidity pools (*exchange / IDB - for some asset classes*).
- Market and limit orders (*not relevant for all asset classes*).
- ...

Papers by Cartea, Jaimungal et al.

- Cartea, Jaimungal, and Ricci. Buy low, sell high: A high frequency trading perspective. SIAM Journal on Financial Mathematics, 2014.
- Cartea, Donnelly, and Jaimungal. Algorithmic trading with model uncertainty. SIAM Journal on Financial Mathematics, 2017.

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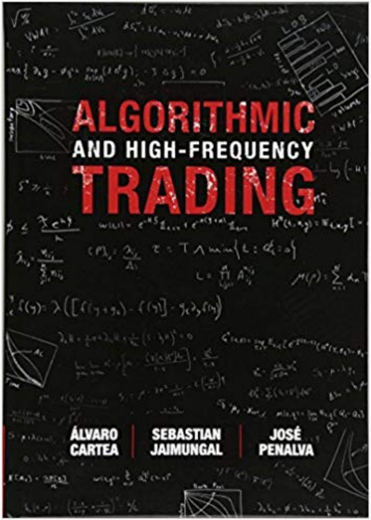

Figure 1: A nice book dealing with market making

Literature

Papers by Guilbaud and Pham

- Guilbaud and Pham. Optimal High-Frequency Trading with limit and market orders. Quantitative Finance, 2013.
- Guilbaud and Pham. Optimal High-Frequency Trading in a Pro-Rata Microstructure with Predictive Information. Mathematical Finance, 2015.

Multi-asset market making

Extensions to multi-asset portfolios

- Guéant. The Financial Mathematics of Market Liquidity. From Optimal Execution to Market Making. CRC Press, 2016.
- Guéant. Optimal market making. Applied Mathematical Finance, 2017.

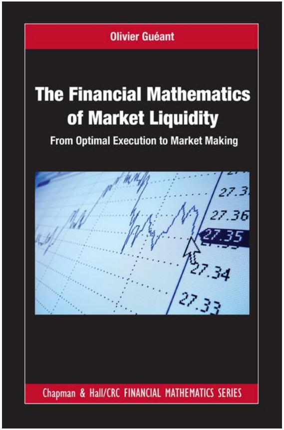

Figure 2: Another nice book

Multi-asset market making

The problem

The number of equations to solve typically grows exponentially with the number of assets.

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- Other works in progress.

Our paper on options is inspired by the first approach.

Asset classes

Extensions to derivatives

- Stoikov and Saglam. Option market making under inventory risk, Review of Derivatives Research, 2009.
- Abergel and El Aoud. A stochastic control approach to option market making. Market Microstructure and Liquidity, 2015.
- **Baldacci, Bergault and Guéant. Algorithmic market making for options. 2020.** *(On ArXiv, in revision)*

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Relevant models should handle several option contracts and tackle the question of Δ-hedging / trading in the underlying asset.

Option market making: the model

The market

Asset price dynamics under $\mathbb{P}$

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S \\ d\nu_t = a_{\mathbb{P}}(t, \nu_t) dt + \xi \sqrt{\nu_t} dW_t^{\nu}. \end{cases}$$

The market

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Asset price dynamics under $\mathbb{Q}$ (pricing measure) - $r = 0$

$$\begin{cases} dS_t = \sqrt{\nu_t} S_t d\widehat{W}_t^S \\ d\nu_t = a_{\mathbb{Q}}(t, \nu_t) dt + \xi \sqrt{\nu_t} d\widehat{W}_t^\nu. \end{cases}$$

The market

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Another one-factor model can be chosen (e.g. Bergomi). Two-factor models are also possible: they increase the dimensionality of the problem by 1.

The market

The options

- We consider $N \geq 1$ European options written on the above asset.
- For $i = 1, \ldots, N$:
  - Maturity of the i-th option: T^i
  - Price process of the i-th option: $(\mathcal{O}_t^i)_{t \in [0, T^i]}$

The market

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Partial differential equation

$\mathcal{O}^i_t = O^i(t, S_t, \nu_t)$ where

$$0 = \partial_t O^i(t,S,\nu) + a_{\mathbb{Q}}(t,\nu)\partial_\nu O^i(t,S,\nu) + \frac{1}{2}\nu S^2 \partial^2_{SS} O^i(t,S,\nu) + \rho\xi\nu S \partial^2_{\nu S} O^i(t,S,\nu) + \frac{1}{2}\xi^2 \nu \partial^2_{\nu\nu} O^i(t,S,\nu).$$

Requests

Distribution of requests

- Requests on option i arrive with intensities $\lambda^{i,b}_{\text{request}}$ and $\lambda^{i,a}_{\text{request}}$.
- Request sizes for option i are distributed according to probability measures $\mu^{i,b}(dz)$ and $\mu^{i,a}(dz)$.
- Bid and ask prices answered for option i (transaction of size z) if the transaction does not violate risk limits:

$$\mathcal{O}^i_t - \delta^{i,b}_t(z) \quad \text{and} \quad \mathcal{O}^i_t + \delta^{i,a}_t(z).$$

- Probabilities of trading:

$$f^{i,b}(\delta^{i,b}_t(z)) \quad \text{and} \quad f^{i,a}(\delta^{i,a}_t(z)).$$

Inventory in the options

Resulting dynamics of the inventory process

$$dq^i_t = \int_{\mathbb{R}^*_+} zN^{i,b}(dt, dz) - \int_{\mathbb{R}^*_+} zN^{i,a}(dt, dz),$$

where $N^{i,b}$ and $N^{i,a}$ are marked point processes with kernels:

$$\nu^{i,b}_t(dz) = \underbrace{\lambda^{i,b}_{\text{request}} f^{i,b}(\delta^{i,b}_t(z))}_{\Lambda^{i,b}(\delta^{i,b}_t(z))} \mathbb{1}_{\{q_{t-}+ze^i \in \mathcal{Q}\}} \mu^{i,b}(dz),$$

$$\nu^{i,a}_t(dz) = \underbrace{\lambda^{i,a}_{\text{request}} f^{i,a}(\delta^{i,a}_t(z))}_{\Lambda^{i,a}(\delta^{i,a}_t(z))} \mathbb{1}_{\{q_{t-}-ze^i \in \mathcal{Q}\}} \mu^{i,a}(dz).$$

Inventory in the underlying asset

Δ-hedging

The market maker ensures perfect Δ-hedging where

$$\Delta_t = \sum_{i=1}^{N} \partial_S \mathcal{O}^i(t, S_t, \nu_t) q_t^i.$$

Inventory in the underlying asset

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Continuous trading is our real assumption:

- The assumption of perfect Δ-hedging can in fact be relaxed.
- One can hedge part of the vega by trading the underlying asset.
  $\rightarrow$ See the appendix of our paper on ArXiv.

Cash dynamics and Mark-to-Market value of the portfolio

Cash dynamics

$$
\begin{aligned}
dX_t \quad = \quad & \sum_{i=1}^{N}\bigg(\int_{\mathbb{R}_+^*} z\Big(\delta_t^{i,b}(z)N^{i,b}(dt,dz) + \delta_t^{i,a}(z)N^{i,a}(dt,dz)\Big) - \mathcal{O}_t^i dq_t^i\bigg) \\
& + S_t d\Delta_t + d\big\langle \Delta, S\big\rangle_t .
\end{aligned}
$$

Cash dynamics and Mark-to-Market value of the portfolio

Cash dynamics

$$dX_t = \sum_{i=1}^{N}\Big(\int_{\mathbb{R}_+^*} z\Big(\delta_t^{i,b}(z)N^{i,b}(dt,dz) + \delta_t^{i,a}(z)N^{i,a}(dt,dz)\Big) - \mathcal{O}_t^i dq_t^i\Big)$$
$$+S_t d\Delta_t + d\langle \Delta, S\rangle_t.$$

Mark-to-Market value of the portfolio

$$V_t = X_t - \Delta_t S_t + \sum_{i=1}^{N} q_t^i \mathcal{O}_t^i.$$

Dynamics of the Mark-to-Market value of the portfolio

Dynamics of the MtM value

$$dV_t = \sum_{i=1}^{N} \int_{\mathbb{R}_+^*} z\Big(\delta_t^{i,b}(z)N^{i,b}(dt,dz) + \delta_t^{i,a}(z)N^{i,a}(dt,dz)\Big) + \sum_{i=1}^{N} q_t^i d\mathcal{O}_t^i - \Delta_t dS_t$$

Dynamics of the Mark-to-Market value of the portfolio

Dynamics of the MtM value

$$\begin{aligned} dV_t &= \sum_{i=1}^{N} \int_{\mathbb{R}_+^*} z\Big(\delta_t^{i,b}(z) N^{i,b}(dt,dz) + \delta_t^{i,a}(z) N^{i,a}(dt,dz)\Big) \\ &\quad + \sum_{i=1}^{N} q_t^i d\mathcal{O}_t^i - \Delta_t dS_t \\ &= \sum_{i=1}^{N} \int_{\mathbb{R}_+^*} z\Big(\delta_t^{i,a}(z) N^{i,a}(dt,dz) + \delta_t^{i,b}(z) N^{i,b}(dt,dz)\Big) \\ &\quad + \sum_{i=1}^{N} q_t^i \partial_\nu O^i(t,S_t,\nu_t)\big(a_{\mathbb{P}}(t,\nu_t) - a_{\mathbb{Q}}(t,\nu_t)\big)dt \\ &\quad + \sqrt{\nu_t}\xi q_t^i \partial_\nu O^i(t,S_t,\nu_t) dW_t^\nu. \end{aligned}$$

Introducing vegas

Vega of the *i*-th option

$$\mathcal{V}_t^i := \partial_{\sqrt{\nu}} O^i(t, S_t, \nu_t) = 2\sqrt{\nu_t}\partial_\nu O^i(t, S_t, \nu_t).$$

Introducing vegas

Vega of the i-th option

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Simplified dynamics of the MtM value

$$\begin{aligned} dV_t &= \sum_{i=1}^N \int_{\mathbb{R}_+^*} z\Big(\delta_t^{i,a}(z)N^{i,a}(dt,dz) + \delta_t^{i,b}(z)N^{i,b}(dt,dz)\Big) \\ &\quad + \sum_{i=1}^N q_t^i \mathcal{V}_t^i \frac{a_{\mathbb{P}}(t,\nu_t) - a_{\mathbb{Q}}(t,\nu_t)}{2\sqrt{\nu_t}} dt + \sum_{i=1}^N \frac{\xi}{2} q_t^i \mathcal{V}_t^i dW_t^\nu . \end{aligned}$$

Option market making: optimization problem, assumptions, and approximations

Objective function

Objective function: risk-adjusted expectation

$$\sup_{\delta \in \mathcal{A}} \mathbb{E}\left[V_T - \frac{\gamma}{2} \int_0^T \left(\sum_{i=1}^N \frac{\xi}{2} q_t^i \mathcal{V}_t^i \right)^2 dt \right].$$

for γ a risk aversion parameter and T a time horizon such that $T < \min_i T^i$.

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$$\sup_{\delta\in\mathcal{A}} \mathbb{E}\left[\int_0^T \sum_{i=1}^N \left(\left(\sum_{j=a,b}\int_{\mathbb{R}_+^*} z\delta_t^{i,j}(z)\Lambda^{i,j}(\delta_t^{i,j}(z))\mathbb{1}_{\{q_{t-}\pm_j ze^i\in\mathcal{Q}\}}\mu^{i,j}(dz)\right) + q_t^i\mathcal{V}_t^i \frac{a_{\mathbb{P}}(t,\nu_t) - a_{\mathbb{Q}}(t,\nu_t)}{2\sqrt{\nu_t}}\right) dt - \frac{\gamma\xi^2}{8}\int_0^T \left(\sum_{i=1}^N q_t^i\mathcal{V}_t^i\right)^2 dt\right],$$

where $\pm_b = +$ and $\pm_a = -$.

Value function

$$u : (t, S, \nu, q) \in [0, T] \times \mathbb{R}^{+^2} \times \mathcal{Q} \mapsto u(t, S, \nu, q)$$

given by

$$u(t, S, \nu, q) = \sup_{(\delta_s)_{s\in[t,T]}\in\mathcal{A}_t} \mathbb{E}_{(t,S,\nu,q)}$$

$$\Bigg[\int_t^T \sum_{i=1}^N \Bigg(\Big(\sum_{j=a,b} \int_{\mathbb{R}_+^*} z\delta_s^{i,j}(z)\Lambda^{i,j}(\delta_s^{i,j}(z))\mathbb{1}_{\{q_{s-}\pm_j ze^i\in\mathcal{Q}\}}\mu^{i,j}(dz)\Big)$$

$$+q_s^i\mathcal{V}_s^i \frac{a_{\mathbb{P}}(s,\nu_s) - a_{\mathbb{Q}}(s,\nu_s)}{2\sqrt{\nu_s}}\Bigg) ds - \frac{\gamma\xi^2}{8}\int_t^T \Big(\sum_{i=1}^N q_s^i\mathcal{V}_s^i\Big)^2 ds\Bigg]$$

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The problem is written in (space) dimension $N + 2$: it is a priori untractable!

Beating the curse of dimensionality

Assumption 1

We approximate the vega of each option over $[0, T]$ by its value at time $t = 0$, namely $\mathcal{V}_t^i = \mathcal{V}_0^i =: \mathcal{V}^i, \quad i = 1, \dots, N.$

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Assumption 2

Authorized inventories correspond to vega risk limits:

$$\mathcal{Q} = \left\{ q \in \mathbb{R}^N \middle| \sum_{i=1}^{N} q^i \mathcal{V}^i \in [-\overline{\mathcal{V}}, \overline{\mathcal{V}}] \right\}, \quad \text{with } \overline{\mathcal{V}} \in \mathbb{R}^{+*}.$$

Change of variables

Portfolio vega

$$\mathcal{V}_t^{\pi} := \sum_{i=1}^{N} q_t^i \mathcal{V}^i.$$

Change of variables

Portfolio vega

$$\mathcal{V}_t^\pi := \sum_{i=1}^N q_t^i \mathcal{V}^i.$$

Optimal control problem

$$v\left(t, \nu, \mathcal{V}^\pi\right) = \sup_{(\delta_s)_{s\in[t,T]}\in\mathcal{A}_t} \mathbb{E}_{(t,\nu,\mathcal{V}^\pi)}$$

$$\left[\int_t^T \left(\left(\sum_{i=1}^N \sum_{j=a,b} \int_{\mathbb{R}_+^*} z\delta_s^{i,j}(z)\Lambda^{i,j}(\delta_s^{i,j}(z))\mathbb{1}_{|\mathcal{V}_s^\pi \pm_j z\mathcal{V}^i| \leq \overline{\mathcal{V}}}\mu^{i,j}(dz)\right)\right.\right.$$

$$\left.\left. +\mathcal{V}_s^\pi \frac{a_{\mathbb{P}}(s,\nu_s) - a_{\mathbb{Q}}(s,\nu_s)}{2\sqrt{\nu_s}} - \frac{\gamma\xi^2}{8}\mathcal{V}_s^{\pi\,2}\right) ds\right].$$

Low-dimensional HJB equation

Low-dimensional HJB equation

The associated Hamilton-Jacobi-Bellman equation is:

$$0 = \partial_t v(t,\nu,\mathcal{V}^\pi) + a_{\mathbb{P}}(t,\nu)\partial_\nu v(t,\nu,\mathcal{V}^\pi) + \frac{1}{2}\nu\xi^2\partial^2_{\nu\nu} v(t,\nu,\mathcal{V}^\pi)$$

$$+\mathcal{V}^\pi \frac{a_{\mathbb{P}}(t,\nu) - a_{\mathbb{Q}}(t,\nu)}{2\sqrt{\nu}} - \frac{\gamma\xi^2}{8}\mathcal{V}^{\pi 2}$$

$$+\sum_{i=1}^N \sum_{j=a,b} \int_{\mathbb{R}_+^*} z \mathbb{1}_{|\mathcal{V}^\pi \pm_j z\mathcal{V}^i| \le \overline{\mathcal{V}}} H^{i,j}\left(\frac{v\big(t,\nu,\mathcal{V}^\pi\big) - v\big(t,\nu,\mathcal{V}^\pi \pm_j z\mathcal{V}^i\big)}{z}\right)\mu^{i,j}(dz),$$

with final condition $v(T,\nu,\mathcal{V}^\pi) = 0$, where

$$H^{i,j}(p) := \sup_{\delta^{i,j} \ge \delta_\infty} \Lambda^{i,j}(\delta^{i,j})(\delta^{i,j} - p).$$

Low-dimensional HJB equation

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The associated Hamilton-Jacobi-Bellman equation is:

$$0 = \partial_t v(t,\nu,\mathcal{V}^\pi) + a_{\mathbb{P}}(t,\nu)\partial_\nu v(t,\nu,\mathcal{V}^\pi) + \frac{1}{2}\nu\xi^2\partial^2_{\nu\nu} v(t,\nu,\mathcal{V}^\pi)$$

$$+\mathcal{V}^\pi \frac{a_{\mathbb{P}}(t,\nu) - a_{\mathbb{Q}}(t,\nu)}{2\sqrt{\nu}} - \frac{\gamma\xi^2}{8}\mathcal{V}^{\pi 2}$$

$$+\sum_{i=1}^{N}\sum_{j=a,b}\int_{\mathbb{R}_+^*} z\mathbb{1}_{|\mathcal{V}^\pi \pm_j z\mathcal{V}^i| \le \overline{\mathcal{V}}} H^{i,j}\left(\frac{v(t,\nu,\mathcal{V}^\pi) - v(t,\nu,\mathcal{V}^\pi \pm_j z\mathcal{V}^i)}{z}\right)\mu^{i,j}(dz),$$

with final condition $v(T,\nu,\mathcal{V}^\pi) = 0$, where

$$H^{i,j}(p) := \sup_{\delta^{i,j} \ge \delta_\infty} \Lambda^{i,j}(\delta^{i,j})(\delta^{i,j} - p).$$

This equation in (space) dimension 2 can be solved numerically on a grid with a Euler scheme and linear interpolation.

Optimal quotes

Once the value function is known, the optimal mid-to-bid and ask-to-mid associated with the N options, are given by the following formula:

Optimal quotes

$$\delta_t^{i,j*}(z) = \max\left(\delta_\infty, \left(\Lambda^{i,j}\right)^{-1}\left(-H^{i,j'}\left(\frac{v\left(t,\nu_t,\mathcal{V}_{t-}^{\pi}\right) - v\left(t,\nu_t,\mathcal{V}_{t-}^{\pi} \pm_j z\mathcal{V}^i\right)}{z}\right)\right)\right).$$

Change of variables when $a_{\mathbb{P}} = a_{\mathbb{Q}}$

If $a_{\mathbb{P}} = a_{\mathbb{Q}}$, then $v(t, \nu, \mathcal{V}^\pi) = w(t, \mathcal{V}^\pi)$ where w is solution of the simpler Hamilton-Jacobi-Bellman:

$$0 = \partial_t w(t, \mathcal{V}^\pi) - \frac{\gamma \xi^2}{8} \mathcal{V}^{\pi\,2}$$

$$+ \sum_{i=1}^{N} \sum_{j=a,b} \int_{\mathbb{R}_+^*} z \mathbb{1}_{|\mathcal{V}^\pi \pm_j z \mathcal{V}^i| \leq \overline{\mathcal{V}}} H^{i,j} \left(\frac{w(t, \mathcal{V}^\pi) - w(t, \mathcal{V}^\pi \pm_j z \mathcal{V}^i)}{z} \right) \mu^{i,j}(dz),$$

with final condition $w(T, \mathcal{V}^\pi) = 0$.

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with final condition $w(T, \mathcal{V}^\pi) = 0$.

The problem is now in (space) dimension 1.

Numerical results

Model parameters

- Stock price at time $t = 0$: $S_0 = 10$ €.
- Instantaneous variance at time $t = 0$: $\nu_0 = 0.0225$ year^{-1}.
- Heston model with $a_{\mathbb{P}}(t, \nu) = \kappa_{\mathbb{P}}(\theta_{\mathbb{P}} - \nu)$ where $\kappa_{\mathbb{P}} = 2$ year^{-1} and $\theta_{\mathbb{P}} = 0.04$ year^{-1}, and $a_{\mathbb{Q}}(t, \nu) = \kappa_{\mathbb{Q}}(\theta_{\mathbb{Q}} - \nu)$ where $\kappa_{\mathbb{Q}} = 3$ year^{-1} and $\theta_{\mathbb{Q}} = 0.0225$ year^{-1}.
- Volatility of volatility: $\xi = 0.2$ year^{-1}.
- Spot-variance correlation: $\rho = -0.5$.

Options

Strikes and maturities

$$\mathcal{K} = \{8 \text{ €}, 9 \text{ €}, 10 \text{ €}, 11 \text{ €}, 12 \text{ €}\}$$

$$\mathcal{T} = \{1 \text{ year}, 1.5 \text{ years}, 2 \text{ years}, 3 \text{ years}\}.$$

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Intensities

$$\Lambda^{i,j}(\delta) = \frac{\lambda_{\text{request}}}{1 + e^{\alpha + \frac{\beta}{\mathcal{V}^i}\delta}}, \; i \in \{1, \ldots, N\}, \; j = a, b.$$

where $\lambda_{\text{request}} = 17640 = 252 \times 30 \text{ year}^{-1}$, $\alpha = 0.7$, and $\beta = 150 \text{ year}^{\frac{1}{2}}$.

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Size of transactions

$z^i = \frac{5 \cdot 10^5}{\mathcal{O}_0^i}$ contracts: $\mu^{i,b}$ and $\mu^{i,a}$ are Dirac masses.

Implied volatility surface

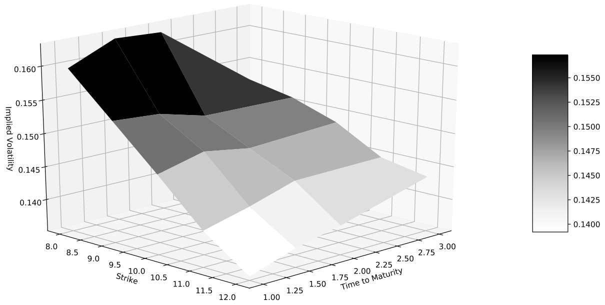

Figure 3: Implied volatility surface associated with the above parameters.

Value function

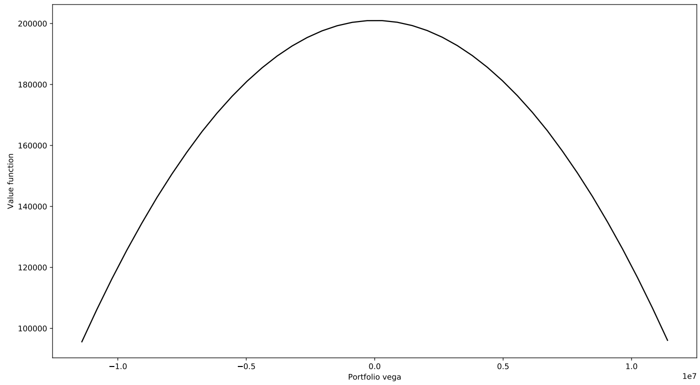

Figure 4: Value function as a function of the portfolio vega for $\nu = 0.0225$ – $\gamma = 10^{-3}$ €$^{-1}$, t=0, T=0.2 days.

Convergence to stationary values

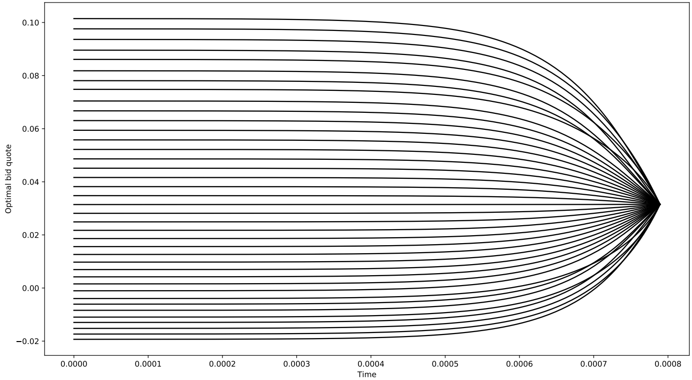

Figure 5: Optimal mid-to-bid quotes as a function of time for $\nu = 0.0225$ – $\gamma = 10^{-3}$ €$^{-1}$.

Optimal bid quotes

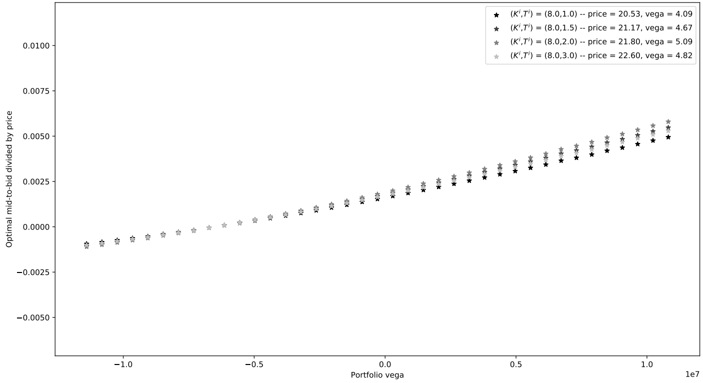

Figure 6: Optimal mid-to-bid quotes divided by option price for $K = 8$ and $\nu = 0.0225$ – $\gamma = 10^{-3}$ €$^{-1}$.

Optimal bid quotes

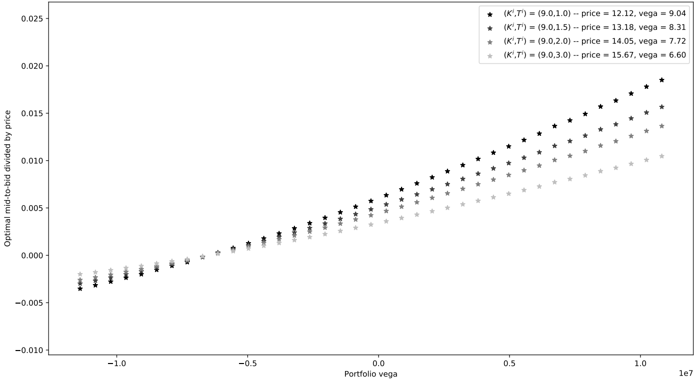

Figure 7: Optimal mid-to-bid quotes divided by option price for $K = 9$ and $\nu = 0.0225$ – $\gamma = 10^{-3}$ €$^{-1}$.

Optimal bid quotes

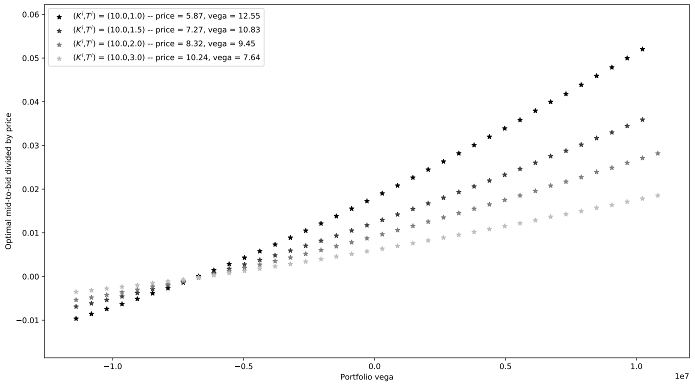

Figure 8: Optimal mid-to-bid quotes divided by option price for $K = 10$ and $\nu = 0.0225$ – $\gamma = 10^{-3}$ €$^{-1}$.

Optimal bid quotes

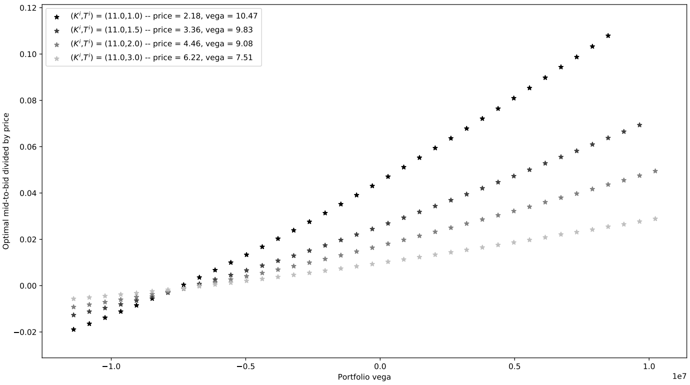

Figure 9: Optimal mid-to-bid quotes divided by option price for $K = 11$ and $\nu = 0.0225$ – $\gamma = 10^{-3}$ €$^{-1}$.

Optimal bid quotes

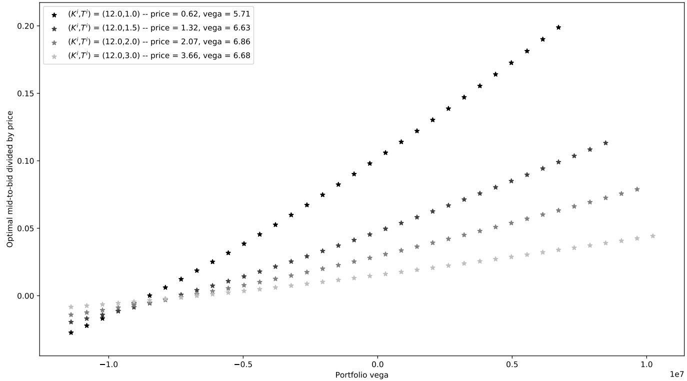

Figure 10: Optimal mid-to-bid quotes divided by option price for $K = 12$ and $\nu = 0.0225$ – $\gamma = 10^{-3}$ €$^{-1}$.

Optimal bid quotes in terms of IV

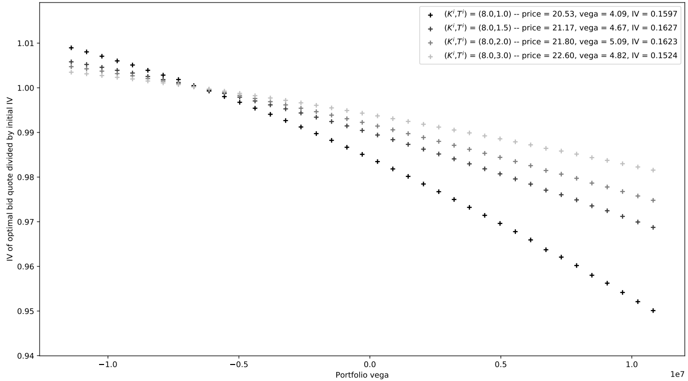

Figure 11: Optimal bid quotes in terms of implied volatility for $K = 8$ and $\nu = 0.0225$ – $\gamma = 10^{-3}$ €$^{-1}$.

Optimal bid quotes in terms of IV

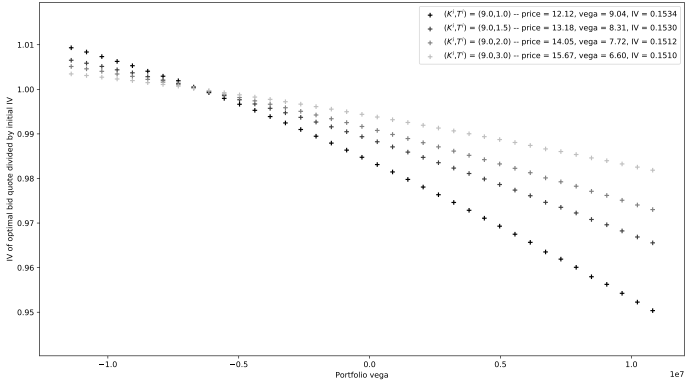

Figure 12: Optimal bid quotes in terms of implied volatility for $K = 9$ and $\nu = 0.0225$ – $\gamma = 10^{-3}$ €$^{-1}$.

Optimal bid quotes in terms of IV

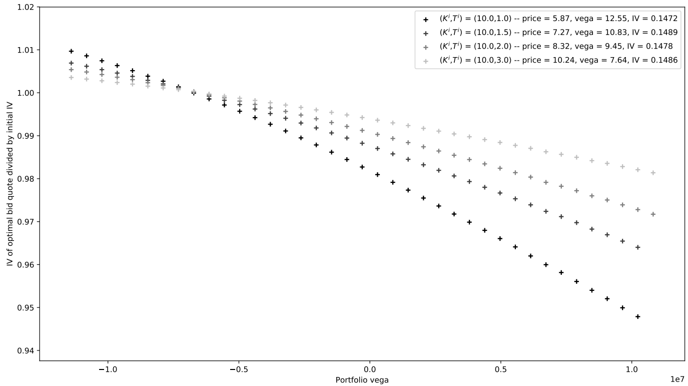

Figure 13: Optimal bid quotes in terms of implied volatility for $K = 10$ and $\nu = 0.0225 - \gamma = 10^{-3}$ €$^{-1}$.

Optimal bid quotes in terms of IV

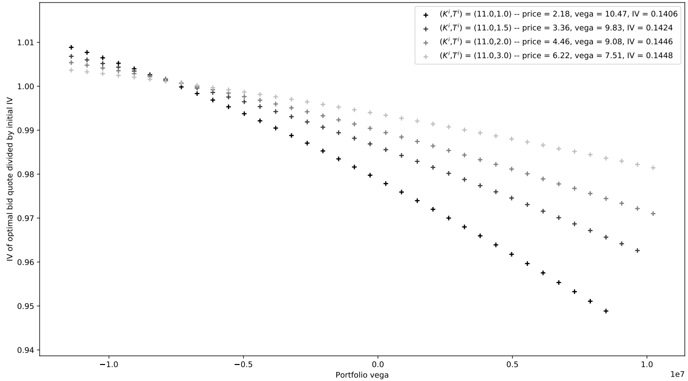

Figure 14: Optimal bid quotes in terms of implied volatility for $K = 11$ and $\nu = 0.0225$ – $\gamma = 10^{-3}$ €$^{-1}$.

Optimal bid quotes in terms of IV

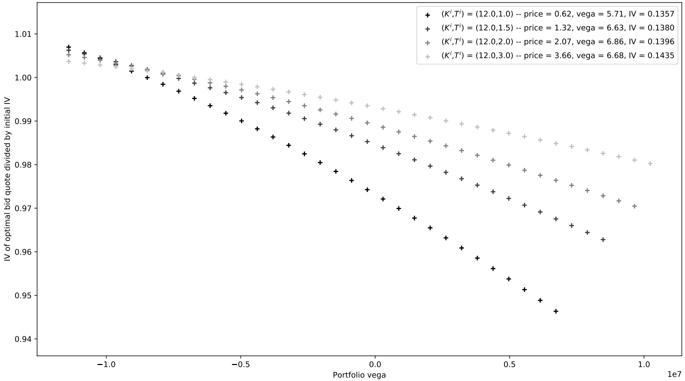

Figure 15: Optimal bid quotes in terms of implied volatility for $K = 12$ and $\nu = 0.0225$ – $\gamma = 10^{-3}$ €$^{-1}$.

Conclusive remarks

- Option market making is tractable using one- or two-factor stochastic volatility models.

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- It is possible to go beyond the constant-vega approximation using a Taylor expansion around that approximation
  $\rightarrow$ see the appendix of our paper.
- A model with several underlying assets can easily be written. The feasibility of numerical approximation with grids depend on the global number of factors.

Questions

Thanks for your attention.
Do not hesitate to make remarks and ask questions.