Exploring Steiner Chains with Möbius Transformations*

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In this article, we use circular gardens, Steiner's porism and Möbius transformations to construct Steiner chains of tangential circles. We then explore some interesting area optimisation problems and touch on Soddy's hexlet and the Duplin cyclide.

1 Introduction

Steiner chains are a beautiful example in circle geometry. A *Steiner chain* is defined as a chain of n circles, each tangent to the previous one and the next one, and also to two given non-intersecting circles [1], which we will call *bounding* circles. We focus exclusively on Steiner chains, one of whose bounding circles lies within the other.

In order to introduce one of the problems that occupied Jacob Steiner in the 19th century, and make it more picturesque, let us suppose the following scenario: a person owns a circular garden with a circular pond in it on one side. Further, this person wants to partially pave the space outside the pond with a chain of touching circular tiles which also touch the pond's border and the circular fence of the garden. He starts outlining the shapes of the tiles on the ground. Eventually, the last drawn tile happens to touch the first one forming a closed chain of touching circles. He then leaves to get some tools to start laying the actual tiles on the ground. Meanwhile it rains and all marks have been erased. When the man comes back, he cannot remember where he started.

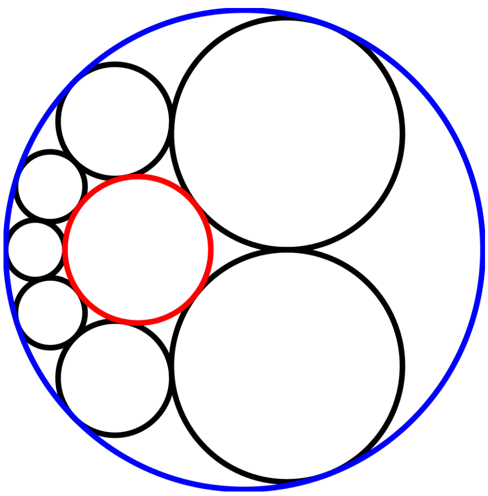

Figure 1: A simple closed Steiner chain with 7 circles.[1]

So, the following natural question arises: if he starts drawing circles at another location, can he hope to eventually obtain a closed chain again? The answer turns out to be affirmative and constitutes what is known as Steiner's porism, presented in the next section. A *porism* is a mathematical proposition, which nowadays usually refers to a statement that is either not true, or is true and holds for an infinite number of values, provided a certain condition is satisfied. We define a *closed* Steiner chain to be such that the first and last circles of the chain are tangent to each other.

For the sake of simplicity, we will limit ourselves to *simple* closed chains, i.e., wrapping only once around the inner bounding circle. In Figure 1, we show a simple closed Steiner chain, consisting of $n = 7$ circles.

A lot of fascinating properties have been discovered so far. For example, it is known that the centres of the circles in the chain lie either on an ellipse (or circle) when one of the bounding circles lies within the other, or on a hyperbola if not. Also, the points of tangency between the circles in the chain happen to lie on a circle [1]. More interestingly, using inversion, a feasibility criterion has been established in [1] for whether a closed Steiner chain is supported for a given n and a pair of bounding circles.

The problem we aim to tackle in this article, is somewhat the opposite: given n positive numbers, does there exist a pair of bounding circles such that we can arrange n circles with radii equal to these numbers in a simple closed Steiner chain between these bounding circles? In addition, if such chains exists, how can we construct them? This can also be reformulated as a geometrical problem of inscribing and circumscribing circles around a chain of touching circles with given radii. We will also consider some area optimisation, regarding maximal and minimal area configurations for a given Steiner chain.

In all these, we mainly rely on the concept of *Möbius transformations*. These are conformal (i.e. angle-preserving) maps in the extended complex plane $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. What is really useful about them, is that they map *circlines* (circles or lines) to circlines [2]. In general a Möbius transformation is a map of the form

$$f(z) = \frac{az+b}{cz+d}, \qquad (1)$$

where $z \in \widetilde{\mathbb{C}}$, and $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ (so as to avoid constant maps). We note that translation, scaling, and inversion are particular cases of Möbius transformations.

2 Steiner's porism

We now state Steiner's porism [1] and give an outline proof of it.

Theorem 1. (Steiner's Porism) Given two bounding circles, if a closed Steiner chain exists, then any circle touching both bounding circles is a member of some closed Steiner chain. In other words, there are infinitely many Steiner chains, which essentially differ by rotation of the starting point of the chain.

Proof. We will first assume non-concentric bounding circles, because otherwise the result is trivial (all Steiner chains are formed by a plain rotation of the first one). Suppose we have the existing Steiner chain, as assumed in the theorem. It is a well-known fact that there exists a Möbius transformation which takes two non-concentric circles into a pair of concentric ones [2]. Since this transformation is conformal, the points of tangency between the images are preserved, and since it maps circlines to circlines, the chain of circles between the original bounding circles gets mapped to a Steiner chain of congruent circles between the two concentric circles. Now, it is clear that we can rotate the resulting

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chain to still obtain a Steiner chain. Inverting the transformation, which is another Möbius transformation, we get a Steiner chain which is dislocated relative to the original one. A little more analysis shows that, in fact, any circle touching the bounding ones gives a Steiner chain in this way.

3 Criteria for existence, and construction of a Steiner chain given n circles.

Suppose now that our gardener has n circular tiles of given radii. Is it possible to create a circular garden with a circular pond in it such that the given tiles can form a simple closed Steiner chain between them? We pursue the answer to this question for $n = 4$, and state how this can be generalised for bigger n. Note that the case $n = 3$ always gives a positive answer, since we can uniquely inscribe and circumscribe a circle between three touching circles, thus, forming a Steiner chain.

Suppose the given radii are $r_k, k = 1, \dots, 4$, assuming r_1 is the smallest. We start with a general Steiner chain S, consisting of 4 circles of radius ρ between two concentric bounding circles of radii r and R (see Figure 2). Our method then consists of finding a Möbius transformation which maps S to another Steiner chain Σ, consisting of 4 circles with the given radii. Eventually, we will obtain a criterion for when this is possible and the map will give us a way to construct the desired Steiner chain.

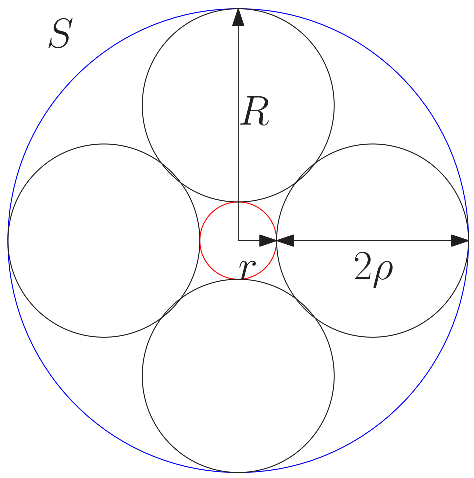

Figure 2: A closed Steiner chain with 4 circles between two concentric bounding circles.

In order for S to be a valid chain, there is a relationship between the various radii that needs to be satisfied. This is a particular case of the general feasibility criterion, mentioned in Section 1.

$$r = (\sqrt{2} - 1)\rho, \qquad R = (\sqrt{2} + 1)\rho. \tag{2}$$

Now, suppose the Möbius transformation which takes S to Σ is given by

$$w = \frac{az + b}{cz + d} \Leftrightarrow z = \frac{dw - b}{a - cw}, \tag{3}$$

where $ad - bc \neq 0$ and without loss of generality, we can assume $d \in \mathbb{R}$ and $d > 0$ (by multiplying top and bottom of w by $\bar{d}$, for example). We further insist that $|d/c| > R$, which ensures that S will be mapped sensibly to Σ, preserving the same configuration. Basically, d/c is the number that is sent to infinity by w, and we want this number to be outside S.

We can now write down the images of the four circles in the chain S as

$$|z - \sqrt{2}\rho e^{i\theta_k}| = \rho \quad \Leftrightarrow \tag{4}$$

$$\left| w - \frac{b + \sqrt{2}\rho a e^{i\theta_k}}{d + \sqrt{2}\rho c e^{i\theta_k}} \right| = \frac{\rho|c|}{d + \sqrt{2}\rho c e^{i\theta_k}} \left| w - \frac{a}{c} \right|, \tag{5}$$

where $\theta_k = \pi k/2, k = 0, \dots, 3$. Similarly, we can write the images of the bounding circles $|z| = r$ and $|z| = R$. Now, we have the images of the four circles, written in an Apollonius form. There are expressions for the radii and centres of such circles [2]. Thus, we need

$$r_k = \frac{|\hat{a}d - b\hat{c}|}{|d + \sqrt{2}\hat{c}e^{i\theta_k}|^2 - |\hat{c}|^2}, \qquad k = 1, \dots, 4, \tag{6}$$

where $\hat{a} = \rho a, \hat{c} = \rho c$, and we note that due to the assumptions on a, b, c, d, the denominator is positive (by a simple reverse triangle inequality). Now, having the four relations (6), it turns out they are more easily manipulated by introducing $\lambda_k = r_k/r_1$. We note that $\lambda_1 = 1$ and $\lambda_k \geq 1$ for $k = 2, 3, 4$. With this, we are able to derive our criteria for existence of a Steiner chain as

$$\lambda_4 = \frac{\lambda_3\lambda_2}{\lambda_3\lambda_2 + \lambda_2 - \lambda_3} \tag{7}$$

$$\frac{2\lambda_3}{(\sqrt{\lambda_3} + 1)^2} < \lambda_2 < \frac{2\lambda_3}{(\sqrt{\lambda_3} - 1)^2} \tag{8}$$

$$\lambda_2 \leq \lambda_3. \tag{9}$$

We note that this also implies $\lambda_4 \leq \lambda_3$. During the derivation process, we have also been able to find relations between the coefficients a, b, c, d, which give us families of Möbius transformations that can construct the required Steiner chain. By way of example and simple check, we use the following radii: $\sqrt{2}/17$, $\sqrt{2}/9$, $\sqrt{2}$, $\sqrt{2}/9$. The corresponding λ_k are: 1, 17/9, 17, 17/9. Thus, it is clear that these do satisfy our criteria, so they can form a Steiner chain in that order. Doing the necessary calculations, we see that one map that satisfies the requirements is $1/(z-2)$, using $\rho = 1/\sqrt{2}$. In order to construct the Steiner chain, we apply this map to S with the specified ρ. We show the result in Figure 3. Our method can be easily generalised for n circles. The difference comes in the relations (6), where now we will have n of them with $\theta_k = 2\pi k/n, k = 0, \dots, n-1$.

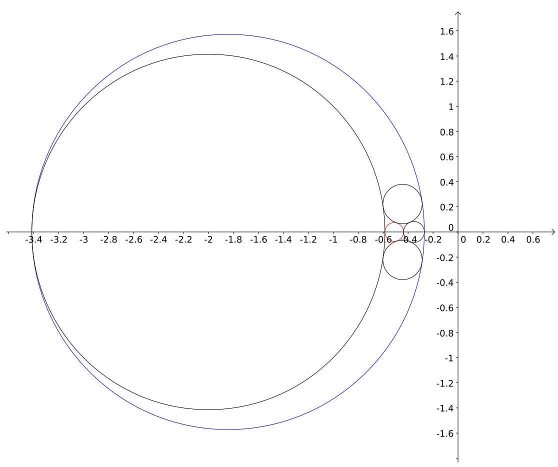

Figure 3: Construction of a Steiner chain with circles' radii $r_1, \dots, r_4$.

4 Area optimisation

Going back to the gardener's scenario, suppose he wants to pave his garden between the fence and the pond with a Steiner chain, using as little concrete for the tiles as possible. More generally, taking into account Steiner's porism, we can ask for the maximal and minimal configurations of a given Steiner chain. Again, this can be done for n circles, but in the interest of brevity, we consider the case $n = 4$. We approach the problem in the following way: starting with a given Steiner chain Σ, we find a Möbius transformation which takes Σ to a Steiner chain S enclosed between concentric bounding circles. Then, we know that all possible Steiner chains in the original configuration are obtained by rotating S by an arbitrary angle θ, and then mapping back using the inverse transformation. Thus, we will express the total area as a function of θ. The expression could subsequently be differentiated to obtain the maximum and minimum, which must exist since the expression for the area is a continuous function over a compact interval $[0, 2\pi]$. Using the previous example, we rotate S (as in Figure 2, using $\rho = 1/\sqrt{2}$) by an angle θ and find the images of the circles in the chain

$$\left| z - \mathrm{e}^{\mathrm{i}(\theta_k + \theta)} \right| = \frac{1}{\sqrt{2}}, \tag{10}$$

using $w = 1/(z-2)$. As a result, we find that the area is

$$A(\theta) = 2\pi \left(\frac{1}{(9+8\cos\theta)^2} + \frac{1}{(9-8\cos\theta)^2} + \frac{1}{(9-8\sin\theta)^2} + \frac{1}{(9+8\sin\theta)^2} \right). \tag{11}$$

Differentiating this, it is straightforward to see that this function has maxima at $\theta = 0, \pi/2, \pi, 3\pi/2$ of value $A \approx 6.46$, and minima at $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ of value $A \approx 1.18$. Thus, the Steiner chain with maximal area is actually the one shown in Figure 3. We show the resulting minimal configuration in Figure 4. See [3] for an interesting animation of rotating Steiner chains for any number of circles.

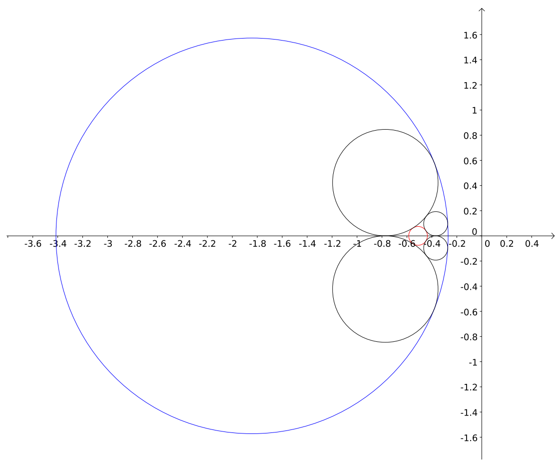

Figure 4: Minimal area configuration for Σ.

5 Results and discussion

Looking at criteria (7–9), we observe several interesting properties for 4-circle Steiner chains. Firstly, if there are unique smallest and biggest circles, then they cannot be neighbours (follows from $\lambda_3 \geq \lambda_2, \lambda_4 \geq \lambda_1$). Second, if the two largest (or smallest) circles are the same, then the other two must also be the same. Finally, if the middle (in size) two circles are the same (i.e., $\lambda_2 = \lambda_4$), then we see that $\lambda_3 = \lambda_2/(2-\lambda_2)$. This means that we must have $\lambda_2 < 2$, i.e., the middle-sized circles cannot be larger than twice the smallest one. We remark that we obtain families of Möbius transformations, which when applied to the Steiner chain between concentric bounding circles, give us a way of constructing the desired Steiner chain from given circles (if they pass the criteria). From a geometrical point of view, our method gives a feasibility criterion and a construction for inscribing and circumscribing a circle around a chain of n pairwise touching circles with specified radii.

6 Conclusions and generalisation

Unlike the classical feasibility problem for Steiner chains given two bounding circles, we have derived criteria for when 4 given circles can form a Steiner chain, regardless of the bounding circles. We have produced a method for constructing such Steiner chains if they exist. This can be generalised to n circles, but the algebra becomes messier. We have also looked at an area optimisation problem and presented a way of determining minimal and maximal area configurations of a given Steiner chain. Steiner chains also have fascinating generalisations to 3D, in terms of spheres, an example of which is Soddy's hexlet, as in Figure 5(a). The interesting thing about it is that, its envelope is the Dupin cyclide (Figure 5(b)), which is the inversion of the torus.

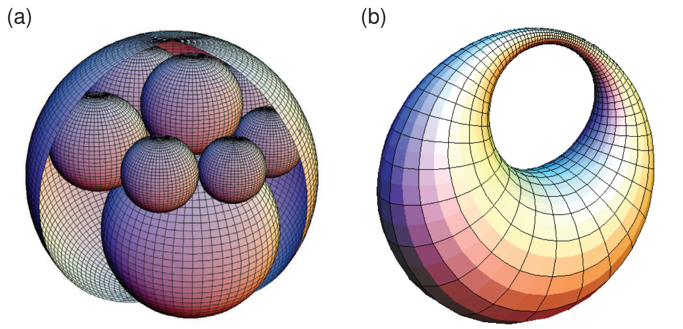

Figure 5: (a) Soddy's hexlet [2] and (b) Dupin cyclide.[3]

Notes

1. From https://commons.wikimedia.org/wiki/File:Steiner_chain_7mer.svg, created by WillowW, under CC Attribution-Share Alike 3.0 Unported license.
2. Created by Horibe Kazunori, http://horibe.jp/Gr8F.HTM.
3. From https://commons.wikimedia.org/wiki/File:Cyclide.png, created by Xah Lee, under CC Attribution-Share Alike 3.0 Unported license.

REFERENCES

1 Pedoe, D. (1970) *Geometry, a Comprehensive Course*, Cambridge University Press, Cambridge.

2 Priestley, H.A. (2003) *Introduction to Complex Analysis*, Oxford University Press, Oxford.

3 Dutch, S. (2012) *Steiner circles*, www.uwgb.edu/dutchs/Geometry/HTMLCanvas/Steiner5.htm (accessed 11 July 2017).